Measurement of acceleration

Acceleration measurement is closely related to the measurement of force. Effect of acceleration on a mass is to give rise to a force. This force is directly proportional to the mass, which if known, will give the acceleration when the force is divided by it.

Characteristics of a spring - mass - damper system

Consider the dynamics of the system shown in Figure (1). In vibration measurement the vibrating table executes vibrations in the vertical direction and may be represented by a complex wave form.

\[ M \frac{d^2 x_2}{dt^2} = K(x_1 - x_2) + c \left( \frac{dx_1}{dt} - \frac{dx_2}{dt} \right) \]  

Dividing through Equation (1) by \( M \) and rearranging we get

\[ \frac{d^2 x_2}{dt^2} + \frac{c}{M} \frac{dx_2}{dt} + \frac{K}{M} x_2 = \frac{c}{M} \frac{dx_1}{dt} + \frac{K}{M} x_1 \]  

The frequency response of the second order system can be represented as

\[ M(\omega) = \frac{x_2 - x_1}{x_{1,0}} = \frac{\left( \frac{\omega}{\omega_n} \right)^2}{\sqrt{\left[ 1 - \left( \frac{\omega}{\omega_n} \right)^2 \right]^2 + \left[ \frac{2\zeta \omega}{\omega_n} \right]^2}} \]
The solution presented above is the basic theoretical framework on which vibration measuring devices are built. In case we want to measure the acceleration, the input to be followed is the second derivative with respect to time of the displacement given by:

$$a(t) = \frac{d^2x_1}{dt^2} = -x_{1,0} \omega^2 \cos(\omega t)$$  \hspace{1cm} (5)

Where

$$x_1 = x_{1,0} \cos(\omega t)$$

The above is nothing but the acceleration response of the system. Plots help us make suitable conclusions.

**Design of second order system for optimum response**

Consider the case shown in Figure (2). For the chosen damping ratio of 0.7 the amplitude response is less than or equal to one for all input frequencies. It can be notice that the response of the system is good for input frequency much larger than the natural frequency of the system. This indicates that the vibration amplitude is more faithfully given by a spring mass damper device with very small natural frequency, assuming that the frequency response is required at relatively large frequencies.

It can be concluded that:

- **Displacement measurement of a vibrating system is best done with a transducer that has a very small natural frequency coupled with**
damping ratio of about 0.7. The transducer has to be made with a large mass with a soft spring.

• Accelerometer is best designed with a large natural frequency. The transducer should use a small mass with a stiff spring. Damping ratio does not have significant effect on the response.

Fig.(2) Amplitude response of the system with damping ratio of 0.7

The acceleration response of the second order system can be shown in figure(3)
Example

A big seismic instrument is constructed with $M = 100 \text{ kg}$, $c/cc = \zeta = 0.7$ and a spring of spring constant $K = 5000 \text{ N/m}$. Calculate the value of linear acceleration that would produce a displacement of 5 mm on the instrument. What is the frequency ratio $\omega/\omega_n$ such that the displacement ratio is 0.99? What is the useful frequency of operation of this system as an accelerometer?

Solution

Since the displacement $x = 5\text{mm} = 0.005\text{m}$ and the spring constant is $K = 5000 \text{ N/m}$. Therefore the spring force corresponding to the given displacement is

$$F = Kx = 5000 \times 0.005 = 25\text{N}$$

The seismic mass is $M = 100 \text{ kg}$. Hence the linear acceleration is given by
\[ a = \frac{F}{M} = \frac{25}{100} = 0.25 \text{m/s}^2 \]

From the response of the second order system equation (7), the amplitude ratio is

\[ M(\omega) = \frac{x_2 - x_1}{x_{1,0}} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[\frac{2\zeta \omega}{\omega_n}\right]^2}} \]

\[ 0.99 = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[\frac{2\zeta \omega}{\omega_n}\right]^2}} \]

Represent \( \omega/\omega_n \) by the symbol \( y \). The damping ratio has been specified as 0.7. From the response of a second order system given earlier the condition that needs to be satisfied is

\[ 0.99 = \frac{(y)^2}{\sqrt{[1 - y^2]^2 + [2\zeta y]^2}} \]

Each \( y^2 \) can be replaced by \( z \)

\[ 0.99 = \frac{z}{\sqrt{[1 - z]^2 + 4\zeta^2 z}} \]

\[ 0.99 = \frac{z}{\sqrt{[1 - z]^2 + 4 \times 0.7^2 \times z}} \]

\[ \frac{z}{\sqrt{[1 - z]^2 + 4 \times 0.7^2 \times z}} = \frac{z}{\sqrt{[1 - z]^2 + 1.96z}} \]

This equation may be simplified to get the following quadratic equation for \( z \).

\[ 0.0203z^2 + 0.04z - 1 = 0 \]

The quadratic equation has a meaningful solution given by \( z = 6.1022 \).
The natural frequency of the system is given by

\[ \omega_n = \sqrt{\frac{K}{M}} = \sqrt{\frac{5000}{100}} = 7.07 \text{ rad/s} \]

The frequency at which the amplitude ratio is equal to 0.99 is thus given by

\[ \omega = y \times \omega_n = 2.47 \times 7.07 = 17.5 \text{ rad/s} \]

Again we shall assume that the useful frequency is such that the acceleration response is 0.99 at this cut off frequency. We have

\[ \frac{M(\omega)}{\left(\frac{\omega}{\omega_n}\right)^2} = 0.99 = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right] + \left[\frac{2\zeta \omega}{\omega_n}\right]^2}} = \frac{1}{\sqrt{[1 - y^2] + [2\zeta y]^2}} \]

By the same above procedure we can get

\[ y = 0.405 \text{ or } \omega = 0.405 \times 7.07 = 2.86 \text{ rad/s} \]

**Piezoelectric accelerometer**

piezoelectric accelerometers are these devices that use a mass mounted on a piezoceramic as shown in Figure (4). The entire assembly is mounted on the device whose acceleration is to be measured. The ENDEVCO MODEL 7703A-50-100 series of piezo-electric accelerometers are typical of such accelerometers. The diameter of the accelerometer is about 16mm. When the mass is subject to acceleration it applies a force on the piezoelectric material. A charge is developed that may be converted to a potential difference by suitable electronics.
The following relations describe the behavior of a piezoelectric transducer.

\[ q = Fd, \quad q = C\Delta V \quad \text{and} \quad C = k \frac{A}{\delta} \quad (1) \]

Combining all these to get

\[ \Delta V = \frac{d \delta}{k A} F = \frac{d \delta}{k A} Ma \quad (2) \]

In Equation (1) the various symbols have the following meanings:

\( C \) = Capacitance of the piezoelectric element, \( \kappa \) = Dielectric constant, \( d \) = Piezoelectric constant, \( A \) = Area of Piezo ceramic, \( \delta \) = Thickness of Piezo ceramic, \( M \) = Seismic mass, \( q \) = Charge, \( \Delta V \) = Potential difference and \( a \) = Acceleration.

Charge sensitivity is defined as

\[ S_q = \frac{q}{a} \quad (3) \]

which has a typical value of \( 50 \times 10^{-8} \text{Coul/g} \). The acceleration is in units of \( g \), the acceleration due to gravity. The voltage sensitivity is defined as

\[ S_v = \frac{\Delta V}{a} = \frac{d \delta}{k A} M \quad (4) \]

Piezo transducers are supplied by many manufacturers. They have specifications typically shown in Table (1).

Table (1) Typical product data
The laser Doppler vibrometer

It is possible to use the laser Doppler for the remote measurement of vibration. Consider the optical arrangement shown in Figure (5). A laser beam is directed towards the object that is vibrating. The reflected laser radiation is split into two beams that travel different path lengths by the arrangement shown in the figure. The two beams are then combined at the detector. The detector signal contains information about the vibrating object and this may be elucidated using suitable electronics.

---

**Laser Doppler Vibrometer**

| Range | 1000 to 5000 N |
| Charge sensitivity | $4 \times 10^{-12}$ Coul/N |
| Capacitance | $25 \times 10^{-12}$ F |
| Stiffness | $5 \times 10^5$ N/m |
| Resonance frequency with 5 g mounted on top load | 35 kHz |
| Effective seismic mass: | |
| Above Piezoelectric element | 3 g |
| Below Piezoelectric element | 18 g |
| Piezoelectric material | Quartz |
| Transducer housing material | SS 316 |
| Diameter | ~ 18 mm |
| Transducer mounting | Threaded spigot and tapped hole in the body |
| Signal conditioning | Charge amplifier |
| Useful frequency range | ~ 10 kHz |

---

**Fig.(5) Working principle of laser Doppler Vibrometer**

---
Consider the target to vibrate in a direction coinciding with the laser incidence direction. Let the velocity of the target due to its vibratory motion be \( V(t) \). The laser beam that is reflected by the vibrating target is split into two beams by the beam splitter 2. One of these travels to the fixed mirror 1 and reaches the photo detector after reflection off the beam splitter 2. The other beam is reflected by fixed mirror 2 and returns directly to the photo detector. The path lengths covered by the two beams are different. The beat frequency is given by

\[
\dot{f}_{\text{beat}} = \frac{4 \Delta l}{\lambda_c} \int_0^\infty A(\omega) \cos(\omega t - \phi(\omega)) d\omega
\]

Where

\( \Delta l \) is the extra distance covered by the second beam with respect to the first.

We notice that the integral in Equation (5) represents the instantaneous acceleration of the vibrating object. Thus the beat frequency is proportional to the instantaneous acceleration of the vibrating target.

\( t_d = \frac{\Delta l}{c} = \text{time delay of the second beam} \)

\( c = \text{speed of the light} \)

Since the speed of light is very large (i.e. \( 3 \times 10^8 \text{m/s} \)) the time delay is rather a small quantity i.e. \( \frac{\Delta l}{c} \ll 1 \)

**Fiber Optic Accelerometer**

Another interesting recent development is a fiber optic accelerometer. The schematic of the instrument is shown in Figure (6).
In this transducer a laser beam is fed in through a cantilever fiber that vibrates with the probe body that is mounted on a vibrating object. The laser beam emerging out of the cantilever is incident on a fiber optic pair that is rigidly fixed and does not vibrate. The vibrating cantilever modulates the light communicated to the two collecting fibers. The behavior of the fiber optic cantilever is given by the following response function which represents the ratio of the relative displacement and the input acceleration.

\[ H(\omega) = \frac{1}{\omega^2} \frac{\cos(\alpha \sqrt{\omega}) + \cosh(\alpha \sqrt{\omega})}{1 + \cos(\alpha \sqrt{\omega}) \cosh(\alpha \sqrt{\omega})} \]  

(6)

\[ \alpha = \left[ \frac{\rho L^4}{EI} \right]^{1/4} \]  

where \( \rho = \text{linear mass density of fiber} \), \( E = \text{Young's Modulus of Elasticity} \) of the fiber

\( \rho = \text{linear mass density of the fiber} \)

\( E = \text{Yong's modulus of Elasticity of the fiber material} \)

\( L = \text{length of cantilever} \)

\( I = \text{Transverse moment of inertia} \).