University of Babylon / College of Engineering / Mechanical Engineering Dept.

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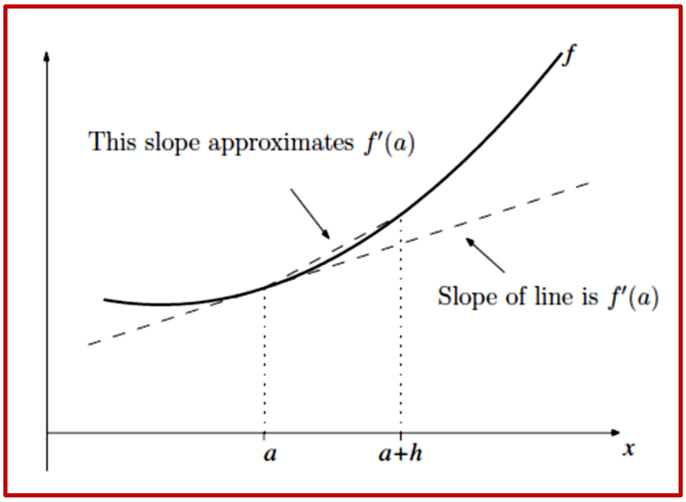
NUMERICAL DIFFERENTIATION & INTEGRATION

Numerical Differentiation

1- Using Taylor Series

-First derivatives

Our aim is to approximate the slope of a curve f at a particular point x = a in terms of f (a) and the value of f at a nearby point where x = a + h.



$$f'(a) \approx \text{slope of short broken line} = \frac{\text{difference in the } y\text{-values}}{\text{difference in the } x\text{-values}} = \frac{f(a+h) - f(a)}{h}.$$

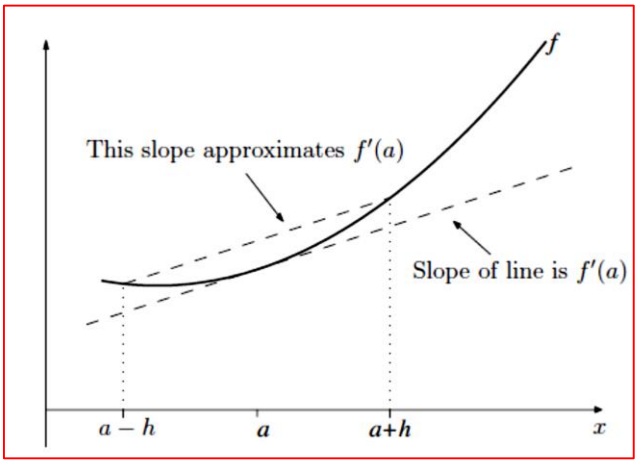
This is called a *forward* difference approximation to the derivative of f.

A second version of this arises on considering a point to the left of a, rather than to the right as we did above. In this case we obtain the approximation

$$f'(a) \approx \frac{f(a) - f(a-h)}{h}$$

This is another difference, called a *backward difference*, approximation to f `(a).

A third method for approximating the first derivative of f can be seen in the next diagram



$$f'(a) \approx \text{slope of short broken line} = \frac{\text{difference in the } y\text{-values}}{\text{difference in the } x\text{-values}} = \frac{f(x+h)-f(x-h)}{2h}$$

This is called a *central difference* approximation to f`` (a).

In practice, the central difference formula is the most accurate.

RELATIVE ERROR

$$|\epsilon_t| = \left|\frac{\textit{exact solution} - \textit{numerical solution}}{\textit{exact solution}}\right| \times 100\%$$

Example

Use a forward difference, and the values of h shown, to approximate the derivative of cos(x) at x = π/3.

(a) h = 0.1 , (b) h = 0.0001

$$f'(a) \approx \frac{\cos(a+h)-\cos(a)}{h} = \frac{0.41104381-0.5}{0.1} = -0.88956192$$

$$f'(a) \approx \frac{\cos(a+h)-\cos(a)}{h} = \frac{0.49991339-0.5}{0.0001} = -0.86605040$$

H.W.

Let f(x) = ln(x) and a = 3. Using both a forward and a central difference, and working to 8 decimal places, approximate f ` (a) using h = 0.1 and h = 0.01.

Example

The distance x of a runner from a fixed point is measured (in meters) at intervals of half a second. The data obtained is

t	0.0	0.5	1.0	1.5	2.0
x	0.00	3.65	6.80	9.90	12.15

Use central differences to approximate the runner's velocity at times t = 0.5s, and t = 1.25s.

Solution

Our aim here is to approximate $x'(t)$. The choice of h is dictated by the available data. Using data with $t = 0.5$s at its centre we obtain

$$x'(0.5) \approx \frac{x(1.0) - x(0.0)}{2 \times 0.5} = 6.80\text{m/s}.$$

Data centred at $t = 1.25$s gives us the approximation

$$x'(1.25) \approx \frac{x(1.5) - x(1.0)}{2 \times 0.25} = 6.20\text{m/s}.$$

Note the value of h used.

H.W.

The velocity v (in m/s) of a rocket measured at half second intervals is

t	0.0	0.5	1.0	1.5	2.0
v	0.000	11.860	26.335	41.075	59.051

Use central differences to approximate the acceleration of the rocket at times t = 1.0s and t = 1.75s.

Second derivatives

A central difference approximation to the second derivative f'' (a) is:

$$f''(a) \approx \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}$$

Example

The distance x of a runner from a fixed point is measured (in meters) at intervals of half a second. The data obtained is

t	0.0	0.5	1.0	1.5	2.0
x	0.00	3.65	6.80	9.90	12.15

Use a central difference to approximate the runner's acceleration at time t =1.5s.

Using data with $t = 1.5$s at its centre we obtain

$$x''(1.5) \approx \frac{x(2.0) - 2x(1.5) + x(1.0)}{0.5^2} = -3.40\text{m/s}^2$$

from which we see that the runner is slowing down.

H.W.

The distance *x*, measured in meters, of a downhill skier from a fixed point is measured at intervals of 0.25 s. The data gathered is

t	0	0.25	0.5	0.75	1	1.25	1.5
x	0	4.3	10.2	17.2	26.2	33.1	39.1

Use a central difference to approximate the skier's velocity and acceleration at the times t =0.25s, 0.75s and 1.25s. Give your answers to 1 decimal place.

2- Using Newton's Forward Difference Formula

$$f(x) \approx P_n(x) = f(x_0) + s\Delta f(x_0) + \frac{s(s-1)}{2!}\Delta^2 f(x_0) + \frac{s(s-1)(s-2)}{3!}\Delta^3 f(x_0)$$
$$\cdots \frac{s(s-1)(s-2)\cdots(s-n+1)}{n!}\Delta^n f(x_0)$$

where

$$x = x_0 + sh$$

We use $p_n(x)$ to calculate the derivatives of f.

That is $f'(x) \simeq p_n'(x)$ for all $x \in [x_0, x_n]$.

For a given x,

$$
\begin{aligned}
f'(x) &\simeq p_n'(x) \\
&= \frac{dp_n}{ds}\frac{ds}{dx} \\
&= \frac{1}{h}\frac{dp_n}{ds}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
f''(x) &\simeq \frac{d^2p_n}{dx^2} \\
&\frac{d}{dx}(\frac{dp_n}{dx}) \\
&= \frac{d}{dx}(\frac{dp_n}{ds}\frac{ds}{dx}) \\
&= \frac{1}{h}\frac{d}{dx}(\frac{dp_n}{ds})
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{h}(\frac{d^2p_n}{ds^2}\frac{1}{h}) \\
&= \frac{1}{h^2}\frac{d^2p_n}{ds^2}
\end{aligned}
$$

Thus in general,

$$
f^{(k)}(x) = \frac{1}{h^k}\frac{d^kp_n}{ds^k}
$$

EXAMPLE

Using Taylor series expansion (forward formula) and Newton forward divided difference, compute first and second derivative at **x = 2** for the following tabulated function

x	1	2	3	4
$f(x)$	2	5	7	10

SOLUTION

x	$f(x)$	Δ	Δ^2	Δ^3
1	2			
		3		
2	5		-1	
		2		2
3	7		1	
		3		
4	10			

Here h = 1

Using Taylor Series

$$f'(2) = \frac{f(2+h)-f(2)}{h} = \frac{f(3)-f(2)}{1} = 2$$

$$f''(2) = \frac{f(2+h)-2f(2)+f(2-h)}{h^2} = \frac{f(3)-2f(2)+f(1)}{1} = -1$$

Using Newton Forward Divided Difference Formula

S =(x-x_o)/h

$$f(x) \approx P_n(x) = f(x_0)+s\Delta f(x_0)+\frac{s(s-1)}{2!}\Delta^2 f(x_0)+\frac{s(s-1)(s-2)}{3!}\Delta^3 f(x_0)$$

$$f'(x) \approx \frac{1}{h}\frac{dp_n}{ds} = \frac{1}{h}\left[\Delta f(x_0) + \frac{2s-1}{2!}\Delta^2 f(x_0) + \frac{3s^2-6s+2}{3!}\Delta^3 f(x_0)\right]$$

Here $x = 2$, $x_0 = 1$, $s = 1$ and $h = 1$

$$f'(2) = 3 - \frac{1}{2} - \frac{1}{3} = 13/6$$

$$f''(x) \approx \frac{1}{h^2}\frac{d^2 p_n}{ds^2} = \frac{1}{h^2}\left[\Delta^2 f(x_0) + (s-1)\Delta^3 f(x_0)\right]$$

$$f''(2) = -1$$

H.W.

Calculate $f^{(4)}(0.15)$

x	0.1	0.2	0.3	0.4	0.5	0.6
$f(X)$	0.425	0.475	0.400	0.450	0.525	0.575

Newton's forward difference formula:

$$p_5(x) = f(x_0) + s\triangle^1 f(x_0) + \frac{s^2 - s}{2}\triangle^2 f(x_0) + \frac{s^3 - 3s^2 + 2s}{6}\triangle^3 f(x_0) +$$

$$+\frac{s^4 - 6s^3 + 11s^2 - 6s}{24}\triangle^4 f(x_0) + \frac{s^5 - 10s^4 + 35s^3 - 50s^2 + 24s}{120}\triangle^5 f(x_0)$$

Differentiating this 4-times we get,

$$\frac{d^4 f}{dx^4} \simeq \frac{dp_5^4}{dx^4} = \frac{1}{h^4}[\triangle^4 f(x_0) + \frac{1}{5}(5s - 10)\triangle^5 f(x_0)$$

$$= \frac{1}{(0.1)^4}[-035 + (0.5 - 2)(0.4)] = -95.00 \times 10^2$$