Week 9: The Transportation Algorithm

1. Introduction

The transportation algorithm follows the *exact steps* of the simplex method , However, instead of using the regular simplex tableau, we take advantage of the special structure of the transportation model to organize the computations in a more convenient form.
Summary of the Transportation Algorithm. The steps of the transportation algorithm are exact parallels of the simplex algorithm.

Step 1. Determine a *starting* basic feasible solution, and go to step 2.
Step 2. Use the optimality condition of the simplex method to determine the *entering variable* from among all the non-basic variables. If the optimality
condition is satisfied, stop. Otherwise, go to step 3.
Step 3. Use the feasibility condition of the simplex method to determine the *leaving variable* from among all the current basic variables, and find the new basic solution. Return to step 2.

2. Determination of the Starting Solution

A general transportation model with m sources and n destinations has $m + n$ constraint equations, one for each source and each destination. However, because the transportation model is always balanced (sum of the supply = sum of the demand), one of these equations is redundant. Thus, the model has $m + n - 1$ independent constraint equations, which means that the starting basic solution consists of $m + n- 1$ basic variables. Thus, in Example below, the starting solution has 3 + 4 - 1 = 6 basic variables.
The special structure of the transportation problem allows securing a non-artificial starting basic solution using one of three methods:

1. Northwest-corner method
2. Least-cost method
3. Vogel approximation method

2.1 Northwest Corner Method

We begin in the upper left corner of the transportation tableau and set $x11$ as large as possible (clearly, $x11$ can be no larger than the smaller of $s1$ and $d1$).

- If $x11=s1$, cross out the first row of the tableau. Also change $d1$ to $d1$-$s1$.
- If $x11=d1$, cross out the first column of the tableau. Change $s1$ to $s1$-$d1$.
- If $x11=s1=d1$, cross out either row 1 or column 1 (but not both!).
  - o If you cross out row, change $d1$ to 0.
  - o If you cross out column, change $s1$ to 0.

Continue applying this procedure to the most northwest cell in the tableau that does not lie in a crossed out row or column.

2.2 **The Least-cost Method**

The least-cost method finds a better starting solution by concentrating on the cheapest routes. The method assigns as much as possible to the cell with the smallest unit cost. Next, the satisfied row or column is crossed out and the amounts of supply and demand are adjusted accordingly.

2.3. **Vogel Approximation Method (VAM)**

V AM is an improved version of the least-cost method that generally, but not always, produces better starting solutions.

Step 1. For each row (column), determine a penalty measure by subtracting the *smallest* unit cost element in the row (column) from the *next smallest* unit cost element in the same row (column).

Step 2. Identify the row or column with the largest penalty. Break ties arbitrarily. Allocate as much as possible to the variable with the least unit cost in the selected row or column. Adjust the supply and demand, and cross out the satisfied row *or* column. If a row and a column are satisfied simultaneously, only one of the two is crossed out, and the remaining row (column) is assigned zero supply (demand).

Step 3.(a) If exactly one row or column with zero supply or demand remains Uncrossed out, stop

(b) If one row (column) with *positive* supply (demand) remains uncrossed out, determine the basic variables in the row (column) by the least-cost method. Stop.

(c) If all the uncrossed out rows and columns have (remaining) zero supply and demand, determine the zero basic variables by the least-cost method. Stop.

(d) Otherwise, go to step 1.

Example :(SunRay Transport)

SunRay Transport Company ships truckloads of grain from three silos to four mills. The supply (in truckloads) and the demand (also in truckloads) together with the unit transportation costs per truckload on the different routes are summarized in the transportation model in Table below. The unit transportation costs, cij, (shown in the northeast corner of each box) are in hundreds of dollars. The model seeks the minimum-cost shipping schedule *xii* between *silo i* and mill *j* (i = *1,2,3;j* = 1,2,3,4).

				supply
10 X11	2 X12	20 X13	11 X14	15
12 X21	7 X22	9 X23	20 X24	25
4 X31	14 X32	16 X33	18 X34	10
demand: 5	15	15	15	

a-**Northwest-Corner Method**

				supply
10 X11	2 X12	20 X13	11 X14	15
12 X21	7 X22	9 X23	20 X24	25
4 X31	14 X32	16 X33	18 X34	10
demand: 5	15	15	15	

10 **5**	2 X12	20 X13	11 X14	15-5=10
12 X21	7 X22	9 X23	20 X24	25
4 X31	14 X32	16 X33	18 X34	10
X	15	15	15	

10 **5**	2 **10**	20 X13	11 X14	x
12 X21	7 X22	9 X23	20 X24	25
4 X31	14 X32	16 X33	18 X34	10
x	15-10=5	15	15	

10 **5**	2 **10**	20 X13	11 X14	x
12 X21	7 **5**	9 X23	20 X24	25-5=20
4 X31	14 X32	16 X33	18 X34	10
x	x	15	15	

10 **5**	2 **10**	20 X13	11 X14	x
12 X21	7 **5**	9 **15**	20 X24	20-15=5
4 X31	14 X32	16 X33	18 X34	10
X	x	x	15	

10 **5**	2 **10**	20 X13	11 X14	x
12 X21	7 **5**	9 **15**	20 **5**	x
4 X31	14 X32	16 X33	18 X34	10
x	x	x	15-5=10	

10 5	2 10	20 X13	11 X14	x
12 X21	7 5	9 15	20 5	x
4 X31	14 X32	16 X33	18 10	10
x	x	x	10	

b-**Least-cost Method**

10 X11	2 X12	20 X13	11 X14	15
12 X21	7 X22	9 X23	20 X24	25
4 X31	14 X32	16 X33	18 X34	10
5	15	15	15	

10 X11	(start) **2** 15	20 X13	11 X14	15-15=0
12 X21	7 X22	9 X23	20 X24	25
4 X31	14 X32	16 X33	18 X34	10
5	x	15	15	

10 X11	(start) **2** 15	20 X13	11 X14	0
12 X21	7 X22	9 X23	20 X24	25
4 5	14 X32	16 X33	18 X34	10-5=5
x	x	15	15	

10 X11	(start) **2** 15	20 X13	11 X14	0
12 X21	7 X22	**9** 15	20 X24	25-15=10
4 5	14 X32	16 X33	18 X34	5
x	x	x	15	

10 X11	(start) **2** 15	20 X13	**11** 0	0
12 X21	7 X22	**9** 15	20 X24	10
4 5	14 X32	16 X33	18 X34	5
x	x	x	15	

10 X11	(start) **2** 15	20 X13	**11** 0	0
12 X21	7 X22	**9** 15	20 X24	10
4 5	14 X32	16 X33	**18** 5	5
x	x	x	15-5=10	

10 X11	(start) **2** 15	20 X13	**11** 0	0
12 X21	7 X22	**9** 15	**20** 10	10
4 5	14 X32	16 X33	**18** 5	5
x	x	x	10	

c-**VAM**

10 X11	2 X12	20 X13	11 X14	15
12 X21	7 X22	9 X23	20 X24	25
4 X31	14 X32	16 X33	18 X34	10
5	15	15	15	

					supply	row-penalty
	10 X11	2 X12	20 X13	11 X14	15	10-2=8
	12 X21	7 X22	9 X23	20 X24	25	9-7=2
	4 5	14 X32	16 X33	18 X34	10	**14-4=10**
Demand	-	15	15	15		
Column penalty	10-4=6	7-2=5	16-9=7	18-11=7		

					supply	row-penalty
	10 X11	**2** 15	20 X13	11 X14	15-15=0	**11-2=9**
	12 X21	7 X22	9 X23	20 X24	25	9-7=2
	4 5	14 X32	16 X33	18 X34	10	16-14=2
Demand	-	15	15	15		
Column penalty		7-2=5	16-9=7	18-11=7		

					supply	row-penalty
	10 X11	**2** 15	20 X13	11 X14	0	20-11=9
	12 X21	7 X22	**9** 15	20 X24	25-15=10	**20-9=11**
	4 5	14 X32	16 X33	18 X34	10	16-14=2
Demand	-	-	15	15		
Column penalty			16-9=7	20-18=2		

After this step only one column stay and we used **step3. Item (b),** Only column 4 is left, and it has a positive supply of 15 units. Applying the least-cost method to that column.

					supply	row-penalty
	10 X11	**2** 15	20 X13	**11** 0	-	
	12 X21	7 X22	**9** 15	20 X24	10	
	4 5	14 X32	16 X33	18 X34	10	
Demand	-	-	-	15		
Column penalty						

					supply	row-penalty
	10 X11	**2** 15	20 X13	11 0	-	
	12 X21	7 X22	9 15	20 10	10	
	4 5	14 X32	16 X33	**18** X34	-	
Demand	-	-	-	15-10=5		
Column penalty						

					supply	row-penalty
	10 X11	**2** 15	20 X13	11 0	-	
	12 X21	7 X22	9 15	**20** 10	-	
	4 5	14 X32	16 X33	18 5	-	
Demand	-	-	-	-		
Column penalty						

3.Iterative Computations of the Transportation Algorithm

After determining the starting solution we use the following algorithm to determine the optimum solution:

Step 1: Use the simplex *optimality condition* to determine the *entering variable* as the current nonbasic variable that can improve the solution. If the optimality condition
is satisfied, stop. Otherwise, go to step 2.

Step 2: Determine the *leaving variable* using the simplex!easibility *condition*. Change the basis, and return to step 1.

Example

Solve the transportation model of Example (SunRay Transport) , starting with the northwest-comer solution which appear in the table below

10 5 ⇒	2 10 ⇓	20 X13	11 X14	X
12 X21	7 5 ⇒	9 15 ⇒	20 5 ⇓	x
4 X31	14 X32	16 X33	18 10	10
x	x	x	10	

And to determine entering variables from among the current non basic variables (those that are not part of the starting basic solution) is done by computing the nonbasic coefficients in the z-row, using the method of multipliers .

In the method of multipliers, we associate the multipliers ui and vi with row i and column *j* of the transportation tableau.

1- For each basic variable compute u and v value as:

ui+vj=cij for each basic xij

Basic variable	(u,v)equation	Solution
X11	u1+v1=10	Set u1=0 → v1=10
X12	u1+v2=2	u1=0 → v2=2
X22	u2+v2=7	v2=2 → u2=5
X23	u2+v3=9	u2=5 → v3=4
X24	u2+v4	u2=5 → v4=15
X34	u3+v4=18	v4=15 → u3=3

2-use u and *v;* to evaluate tile nonbasic variables by computing as:

ui+vi-cij for each nonbasic xij

The results of these evaluations are shown in the following table:

Nonbasic variable	ui+vi-cij
X13	u1+v3-c13=0+4-20=-16
X14	u1+v4-c14=0+15-11=4
X21	u2+v1-c21=5+10-12=3
X31	**u3+v1-c31=3+10-4=9**
X32	u3+v-c32=3+2-14=-9
X33	u3+v3-c33=3+4-16=-9

Because the transportation model seeks to *minimize* cost, the entering variable is the one having the *most positive* coefficient in the z-row. Thus, x31 is the entering variable.

To determine leave variable we applied

1-construct a ***closed loop*** that starts and ends at the entering variable cell (3, 1). The loop consists of ***connected horizontal*** and ***vertical*** segments only (no diagonals are allowed). Except for the entering variable cell, each comer of the closed loop must coincide with a basic variable.

	V 1=10	v2=2	v3=4	v4=15	supply
u1=0	10 5- ϕ	2 10+ ϕ	20 -16	11 4	15
u2=5	12 3	7 5- ϕ	9 15	20 5 + ϕ	15
u3=3	**4** ϕ 9	14 -9	16 -9	18 10- ϕ	10
Demand	5	15	15	15	

2- we assign the amount to the entering variable cell (3, 1). For the supply and demand limits to remain satisfied, we must alternate between subtracting and adding the amount II at the successive *corners* of the loop, For $\phi \geq 0$, the new values of the variables then remain nonnegative if

$$x11=5-\phi \geq 0$$
$$x22=5-\phi \geq 0$$

$$x34=10-\phi \geq 0$$

The corresponding maximum value of ϕ is 5, which occurs when both x11and x22 reach zero level Because only one current basic variable must leave the basic solution, we can choose either x11 Or x22 as the leaving variable and The selection of x31 (= 5) as the entering variable and x11 as the leaving variable.

Second iteration

1-entering variable determine

Basic variable	(u,v)equation	Solution
X12	u1+v2=2	Set u1=0 → v2=2
X22	u2+v2=7	v2=2 → u2=5
X23	u2+v3=9	u2=5 → v3=4
X24	u2+v4=20	u2=5 → v4=15
X31	u3+v1=4	u3=3 → v1=1
X34	u3+v4=18	v4=15 → u3=3

Nonbasic variable	ui+vi-cij
X11	u1+v1-c11=0+1-10=-9
X13	u1+v3-c13=0+4-20=-16
X14	**u1+v4-c14=0+15-11=4**
X21	u2+v1-c21=5+1-12=-6
X32	u3+v2-c32=3+2-14=-9
X33	u3+v3-c33=3+4-16=-9

Entering variable is cell(1,4)

	V 1=1	v2=2	v3=4	v4=15	supply
u1=0	10 -9	2 15- ϕ	20 -16	11 ϕ 4	15
u2=5	12 -6	7 0+ ϕ	9 15	20 10- ϕ	15
u3=3	**4** **5**	14 -9	16 -9	18 5	10
demand	5	15	15	15	

2-leaving variable determine

$$x12=15-\phi \geq 0$$
$$x24=10-\phi \geq 0$$

Given the new basic solution, we repeat the computation of the multipliers" and *v*, as Table below shows. The entering variable is x14. The closed loop shows that x14 = 10 and that the leaving variable is x24. .

	V 1=-3	v2=2	v3=4	v4=11	supply
u1=0	10 -9	2 5	20 -16	11 10	15
u2=5	12 -6	7 10	9 15	20 0	15
u3=7	**4** **5**	14 -9	16 -9	18 5	10
demand	5	15	15	15	

Third iteration
1-entering variable determine

Basic variable	(u,v)equation	solution
X12	u1+v2=2	Set u1=0 → v2=2
X14	u1+v4=11	u1=0 → v4=11
X22	u2+v2=7	v2=2 → u2=5
X23	u2+v3=9	u2=5 →v3=4
X31	u3+v1=4	u3=7 →v1=-3
X34	u3+v4=18	v4=11 →u3=7

Nonbasic variable	ui+vi-cij
X11	u1+v1-c11=0+(-3)-10=-14
X13	u1+v3-c13=0+4-20=-16
X21	u2+v1-c21=5+(-3)-12=-10
X24	u2+v4-c24=5+11-20=-3
X32	u3+v2-c32=7+2-14=-5
X33	u3+v3-c33=7+4-16=-5

All value of nonbasic variable is negative and stop.