Lecture 21
Game Theory :
Games with Mixed Strategies
(analytic and graphic methods)

21.1 Games with Mixed Strategies

In certain cases, no pure strategy solutions exist for the game. In other words, saddle point does not exist. In all such game, both players may adopt an optimal blend of the strategies called **Mixed Strategy** to find a saddle point. The optimal mix for each player may be determined by assigning each strategy a probability of it being chosen. Thus these mixed strategies are probabilistic combinations of available better strategies and these games hence called **Probabilistic games**.

The probabilistic mixed strategy games without saddle points are commonly solved by any of the following methods

Sl. No.	Method	Applicable to
1	Analytical Method	2x2 games
2	Graphical Method	2x2, mx2 and 2xn games
3	Simplex Method	2x2, mx2, 2xn and mxn games

21.1.1 Analytical Method

A 2 x 2 payoff matrix where there is no saddle point can be solved by analytical method.
Given the matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Value of the game is

$$V = \frac{a_{11}\,a_{22} - a_{21}\,a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

With the coordinates

$$x_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} \quad , \quad x_2 = \frac{a_{11} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$y_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} \quad , \quad y_2 = \frac{a_{11} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

<u>Alternative procedure to solve the strategy</u>

- Find the difference of two numbers in column 1 and enter the resultant under column 2. Neglect the negative sign if it occurs.
- Find the difference of two numbers in column 2 and enter the resultant under column 1. Neglect the negative sign if it occurs.
- Repeat the same procedure for the two rows.

1. Solve

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix} \end{array}$$

Solution

It is a 2 x 2 matrix and no saddle point exists. We can solve by analytical method

$$\begin{array}{ccc} & B & \\ A & \begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix} & \begin{matrix} 1 \\ 4 \end{matrix} \\ & \begin{matrix} 3 & \quad 2 \end{matrix} & \end{array}$$

$$V = \frac{a_{11}\, a_{22} - a_{21}\, a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{20 - 3}{9 - 4}$$

V = 17 / 5

$S_A = (x_1, x_2) = (1/5, 4/5)$

$S_B = (y_1, y_2) = (3/5, 2/5)$

2. Solve the given matrix

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} \end{array}$$

Solution

$$\begin{array}{ccc} & B & \\ A & \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} & \begin{matrix} 1 \\ 3 \end{matrix} \\ & \begin{matrix} 1 & \quad 3 \end{matrix} & \end{array}$$

$$V = \frac{a_{11}\, a_{22} - a_{21}\, a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{0 - 1}{2 + 2}$$

V = - 1 / 4

$S_A = (x_1, x_2) = (1/4, 3/4)$

$S_B = (y_1, y_2) = (1/4, 3/4)$

21.1.2 Graphical method

The graphical method is used to solve the games whose payoff matrix has

- Two rows and n columns (2 x n)
- m rows and two columns (m x 2)

Algorithm for solving 2 x n matrix games

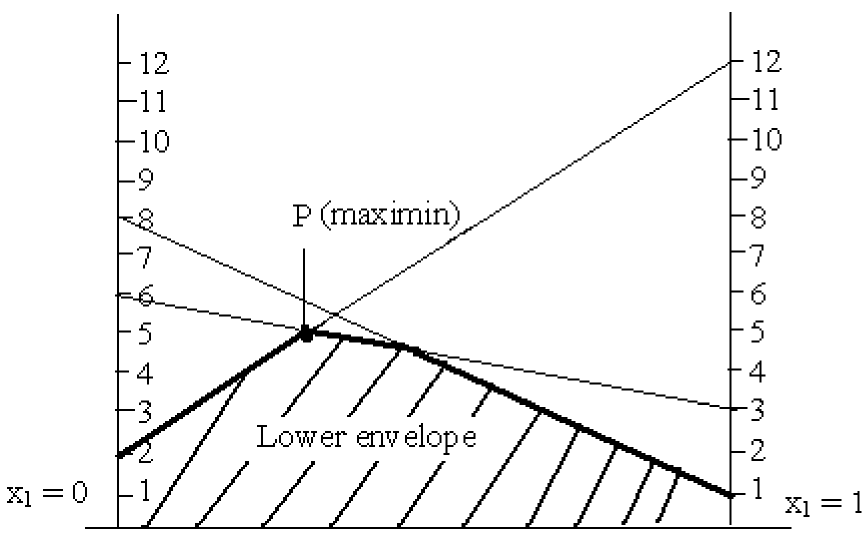
- Draw two vertical axes 1 unit apart. The two lines are $x_1 = 0$, $x_1 = 1$
- Take the points of the first row in the payoff matrix on the vertical line $x_1 = 1$ and the points of the second row in the payoff matrix on the vertical line $x_1 = 0$.
- The point a_{1j} on axis $x_1 = 1$ is then joined to the point a_{2j} on the axis $x_1 = 0$ to give a straight line. Draw 'n' straight lines for j=1, 2… n and determine the highest point of the lower envelope obtained. This will be the **maximin point**.
- The two or more lines passing through the maximin point determines the required 2 x 2 payoff matrix. This in turn gives the optimum solution by making use of analytical method.

Example 1

Solve by graphical method

$$\begin{array}{cc} & \begin{array}{ccc} B1 & B2 & B3 \end{array} \\ \begin{array}{c} A1 \\ A2 \end{array} & \begin{bmatrix} 1 & 3 & 12 \\ 8 & 6 & 2 \end{bmatrix} \end{array}$$

Solution



$$\begin{array}{ccc} & \begin{array}{cc} B2 & B3 \end{array} & \\ \begin{array}{c} A1 \\ A2 \end{array} & \begin{bmatrix} 3 & 12 \\ 6 & 2 \end{bmatrix} & \begin{array}{c} 4 \\ 9 \end{array} \\ & \begin{array}{cc} 10 & 3 \end{array} & \end{array}$$

$$V = \frac{a_{11}\,a_{22} - a_{21}\,a_{12}}{(a_{11}+a_{22}) - (a_{12}+a_{21})} = \frac{6-72}{5-18}$$

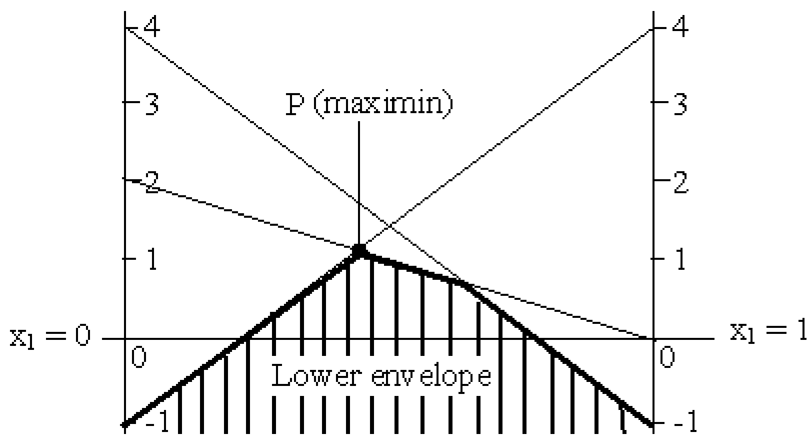
V = 66/13
S_A = (4/13, 9 /13)
S_B = (0, 10/13, 3 /13)

Example 2

Solve by graphical method

$$\begin{array}{c|c} & \begin{array}{ccc} B1 & B2 & B3 \end{array} \\ \begin{array}{c} A1 \\ A2 \end{array} & \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & 2 \end{bmatrix} \end{array}$$

Solution



$$\begin{array}{cccc} & \begin{array}{cc} B1 & B3 \end{array} & \\ \begin{array}{c} A1 \\ A2 \end{array} & \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix} & \begin{array}{c} 3 \\ 4 \end{array} \\ & \begin{array}{cc} 2 & 5 \end{array} & \end{array}$$

$$V = \frac{a_{11}\,a_{22} - a_{21}\,a_{12}}{(a_{11}+a_{22}) - (a_{12}+a_{21})} = \frac{8-0}{6+1}$$

V = 8/7
S_A = (3/7, 4 /7)
S_B = (2/7, 0, 5 /7)

<u>Algorithm for solving m x 2 matrix games</u>

- Draw two vertical axes 1 unit apart. The two lines are x_1 =0, x_1 = 1

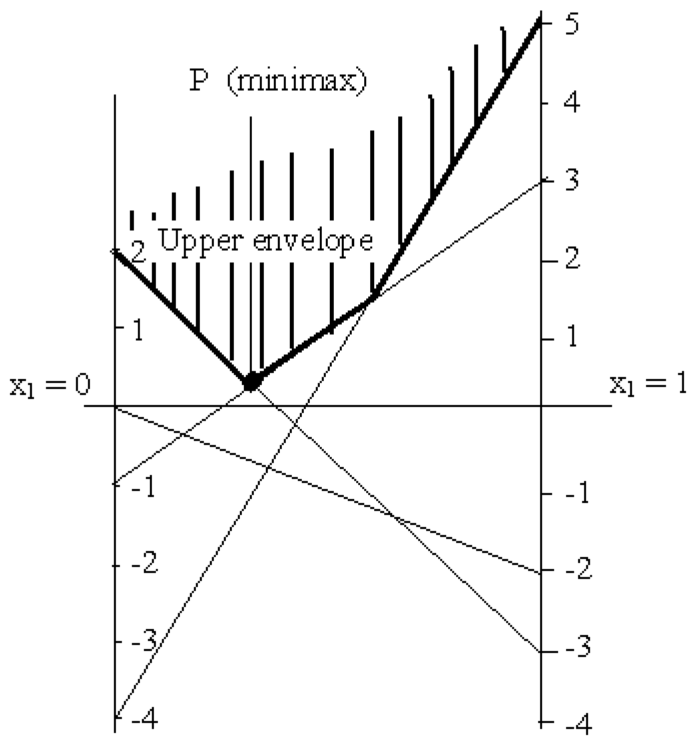
- Take the points of the first row in the payoff matrix on the vertical line $x_1 = 1$ and the points of the second row in the payoff matrix on the vertical line $x_1 = 0$.
- The point a_{1j} on axis $x_1 = 1$ is then joined to the point a_{2j} on the axis $x_1 = 0$ to give a straight line. Draw 'n' straight lines for j=1, 2… n and determine the lowest point of the upper envelope obtained. This will be the **minimax point**.
- The two or more lines passing through the minimax point determines the required 2 x 2 payoff matrix. This in turn gives the optimum solution by making use of analytical method.

Example 1

Solve by graphical method

$$
\begin{array}{cc}
 & \begin{array}{cc} B1 & B2 \end{array} \\
\begin{array}{c} A1 \\ A2 \\ A3 \\ A4 \end{array} & \begin{bmatrix} -2 & 0 \\ 3 & -1 \\ -3 & 2 \\ 5 & -4 \end{bmatrix}
\end{array}
$$

Solution



$$
\begin{array}{ccc}
 & \begin{array}{cc} B1 & B2 \end{array} & \\
\begin{array}{c} A2 \\ A3 \end{array} & \begin{bmatrix} 3 & -1 \\ -3 & 2 \end{bmatrix} & \begin{array}{c} 5 \\ 4 \end{array} \\
 & \begin{array}{cc} 3 & 6 \end{array} &
\end{array}
$$

$$V = \frac{a_{11}\, a_{22} - a_{21}\, a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{6 - 3}{5 + 4}$$

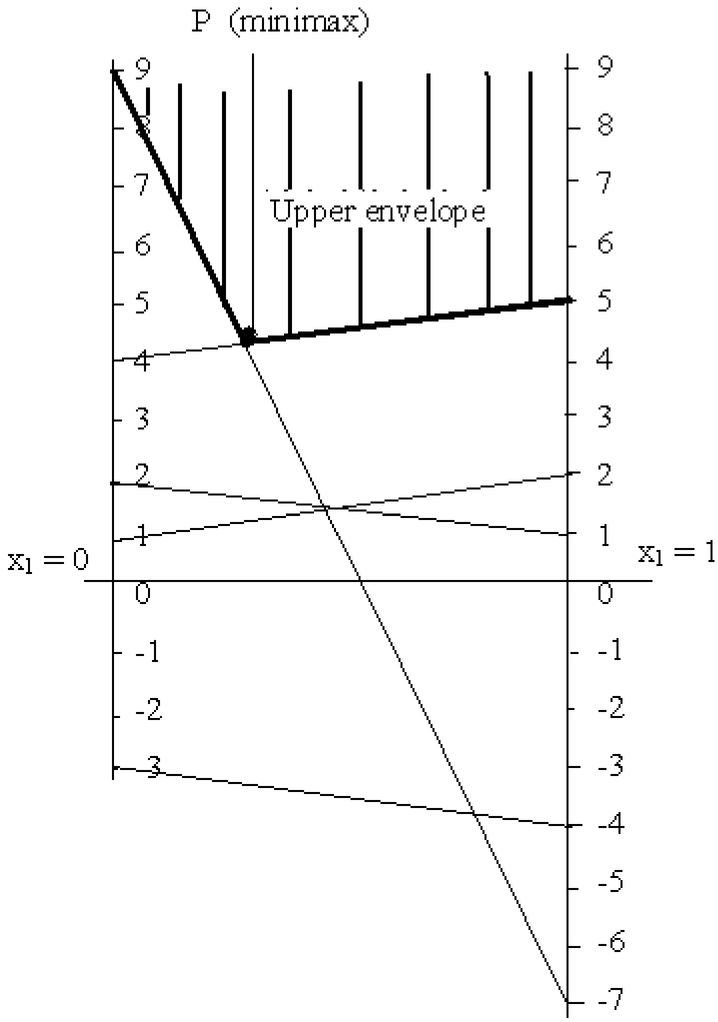
$V = 3/9 = 1/3$
$S_A = (0, 5/9, 4/9, 0)$
$S_B = (3/9, 6/9)$

Example 2

Solve by graphical method

$$\begin{array}{c} \\ A1 \\ A2 \\ A3 \\ A4 \\ A5 \end{array} \begin{array}{c} \begin{array}{cc} B1 & B2 \end{array} \\ \begin{bmatrix} 1 & 2 \\ 5 & 4 \\ -7 & 9 \\ -4 & -3 \\ 2 & 1 \end{bmatrix} \end{array}$$

Solution



$$\begin{array}{cc} & \begin{array}{cc} B1 & B2 \end{array} & \\ \begin{array}{c} A2 \\ A3 \end{array} & \begin{bmatrix} 5 & 4 \\ -7 & 9 \end{bmatrix} & \begin{array}{c} 16 \\ 1 \end{array} \\ & \begin{array}{cc} 5 & 12 \end{array} & \end{array}$$

$$V = \frac{a_{11}\, a_{22} - a_{21}\, a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{45 + 28}{14 + 3}$$

V = 73/17

S_A = (0, 16/17, 1/17, 0, 0)

S_B = (5/17, 12 /17)