CHAPTER 5

INTEGRATION

- Integration is the process of calculating an integral.
- Integral calculus is the mathematics we use to find length, area …. etc.
- By integration we get (general solution) of the problem and by applying (boundary conditions) we get (particular solution). This is for indefinite integral.

5.1 Antiderivatives:

The antiderivative is the function when the original function obtained from its derivative. It is defined by *F*

For example if $f(x) = \cos x$, $F(x) = \sin x$ There

are two types of integration:

1. Definite integral $\int_a^b f(x)dx$

- $\int$: Integral sign (it is elongated S chosen by Leibniz from the letter *S* in German word summation). - Give numerical values - No constant of integrals.

2. Indefinite integral $\int f(x)dx$
- General solution (constant of integration C) -
Particular solution (applying boundary condition)

5.2 Definite integral:

The area problem:

If we have the function $y = f(x) = x^2$, and want to find the area under the graph from $x = 0$ to $x = 1$:

- The area is not regular; it is not easy to find it.
- We can estimate the area by dividing it to small strips.

- If the area is divided into four strips of rectangular shape:

Base = ¼ unit

Height = the right edge of the rectangle.

The area equal to

$R4 = \frac{1}{4}(1/4)^2 + \frac{1}{4}(1/2)^2 + \frac{1}{4}(3/4)^2 + \frac{1}{4}(1)^2$

$= 15/32 = 0.46875$

- If the height is equal to the left edge of the rectangle: $L4 = \frac{1}{4}(0)^2 + \frac{1}{4}(1/4)^2 + \frac{1}{4}(1/2)^2 + \frac{1}{4}(3/4)^2 = 7/32 = 0.21875$

The exact solution is greater than L4

$0.21875 < A$ (exact area) < 0.46875

- If we repeat this procedure with a larger number of rectangular strips, the values of R and L closed to exact area $A = 1/3 = 0.33333333$ obtained from integration.
  For $R_{1000} = 0.3338335$, $L_{1000} = 0.3328335$

Now: if we have the function $y = f(x)$, and we want to find the exact area under the graph of this function from $x = a$ to $x = b$, divide it into n rectangles:

Take a typical rectangle (kth rectangle):

Area of kth rectangle = $f(C_x) \cdot \Delta X_k$

The sum of areas of rectangles (S) is:

$S = \sum_{k=1}^{n} f(C_k).\Delta X_k$

The Greek capital letter $\sum$ (sigma) is used to indicate sums.

The exact area (A) = $\lim_{n\to\infty} \sum_{k=1}^{n} f(C_k).\Delta X_k$

The definite integral of $f(x)$ from $x = a$ to $x = b$ is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(C_k).\Delta X_k$$

Where;

a: Lower limit of integration *b*: Upper

limit of integration *dx*: Differential

(index integration).

Notes:

1. All continuous functions are integrable.
2. If *f(x)* is negative, the area becomes below the *x*-axis.

5.3 The mean value theorem for definite integral:

If f(x) is continuous on the closed interval [a, b], then, at some point c in the interval [a, b]

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

For example: the average value of $y = x^2$ from x = 0 to x = 1 is $\frac{1}{1-0}\left(\frac{1}{3}\right) = \frac{1}{3}$

Rules for definite integral:

1. $\int_a^a f(x)dx = 0$

$\int_a^b f(x)dx = -$

$\int_a^b Kf(x)dx =$

$\int_a^b [f(x) \pm g(x)$

$\int_a^c f(x)dx = \int_a^b$

$\int_b^c f(x)dx = \int_a^c$

2. $\int_b^a f(x)dx$

$K\int_a^b f(x)dx$

$]dx = \int_a^b f$

$f(x)dx +$

$f(x)dx - \int$

3.

4. $(x)dx \pm \int_a^b g(x)dx$

$\int_b^c f(x)dx$

${}_a^b f(x)dx$

5.

6.

7. If $g(x) \geq f(x)$ $\int_a^b g(x)dx \geq \int_a^b f(x)dx$

8. If $f(x) \geq 0$ $\int_a^b f(x)dx \geq 0$

5.4 The fundamental theorems of integral calculus:

5.4.1 The first fundamental theorem:

If f is continuous on [a, b], then the function F(x) =$\int_a^b f(t)dt$ has a derivative at every point on [a, b] and

$$\frac{dF}{dx} = \frac{d}{dx}\int_a^x f(t)dt = f(x)$$

Example 1: Find dy/dx for y =$\int_{-\pi}^{x} \cot t\, dt$

Solution: dy/dx = $\frac{d}{dx}\int_{-\pi}^{x} \cot t\, dt = \cos x$

Example 2: Find dy/dx for $y = \int_1^{x^2} \cos t\ dt$

Solution: Let $u = x^2$ $y = \int_1^{x^2} \cos t\ dt = \int_1^{u} \cos t\ dt$

$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = \cos u\,.\frac{du}{dx} = \cos x^2 .\ 2x = 2x \cos x^2$

5.4.2 The second fundamental theorem (Integral evaluate theorem):

If *f* is continuous at every point on [a, b] and F is any antiderivative of *f* on [a, b] then:

$$\int_a^b f(x)dx = F(b) - F(a)$$

Example 3: calculate $\int_0^\pi \cos x\, dx$

$\int_0^\pi \cos x\, dx = [\sin x]_0^\pi = \sin \pi - \sin 0 = 0 - 0 = 0$

5.5 Integration Formulas:

1. $\int_a^b u^n du = \left.\frac{u^{n+1}}{n+1}\right|_a^b$
2. $\int_a^b du = u|_a^b$
3. $\int_a^b (du + dv) = \int_a^b du + \int_a^b dv$
4. $\int_a^b \sin u\, du = -\cos u|_a^b$
5. $\int_a^b \cos u\, du = \sin u|_a^b$
6. $\int_a^b sec^2\, u\, du = \tan u|_a^b$
7. $\int_a^b \sec u \tan u\, du = \sec u|_a^b$
8. $\int_a^b csc^2\, u\, du = -cot u|_a^b$
9. $\int_a^b \csc u \cot u\, du = -csc\, u|_a^b$

5.6 Area Under Curve

Example 4: Find the area under the curve $y = x + 2$ from $x = 1$ to $x = 4$.

Solution: $A = \int dA = \int_1^4 ydx = \int_1^4 (x+2)dx$

$= \left.\frac{x^2}{2} + 2x\right|_1^4$

$= \left(\frac{16}{2} + 8\right) - \left(\frac{1}{2} + 2\right)$

= 16 – 5/2 = 13.5 units

Example 5: Find the area under the curve $y = cos\ x$ from $x = 0$ to $x = \pi/2$.

Solution: $A = \int_0^{\pi/2} y dx = \int_0^{\pi/2} \cos x\, dx = \sin x|_0^{\pi/2} = \sin \pi/2 - \sin 0 = 1$ units

Example 6: Find the area bounded by the curve $y = f(x) = \sqrt{2x+1}$ and the lines x = 0, x = 4 and the x-axis.

Solution: $A = \int_0^4 (2x+1)^{1/2} dx$

$$= \frac{1}{2}.\frac{2}{3}(2x+1)^{3/2}\Big|_0^4 = \frac{1}{3}[(3^2)^{\frac{3}{2}} - (1)]$$

= 1/3 (26) = 26/3 units

Example 7: Find the area of the region between the x-axis and the curve $y = x^3 - 4x$. $-2 \le x \le 2$ **Solution**:

In this case, the graph has +ve and –ve values of area. We must divided the interval [-2, 2] into [-2, 0] and [0, 2] and take the absolute value of results. $y = 0$ $x^3 - 4x = 0$ $x(x^2 - 4) = 0$ $x(x-2)(x+2) = 0$ $x = 0,\ x = 2,\ x = -2$

$$A_1 = \int_{-2}^0 (x^3 - 4x)\ dx = \left[\frac{x^4}{4} - 2x^2\right]_{-2}^0 = [0] - [4 - 8] = 4$$

$$A_2 = \int_0^2 (x^3 - 4x)\ dx = \left[\frac{x^4}{4} - 2x^2\right]_0^2 = [4 - 8] - [0] = -4$$

$A = |A_1| + |A_2|$

$= |4| + |-4|$

$= 4 + 4 = 8$ units

Example 8: Evaluate $\int_{\frac{\pi^2}{4}}^{\pi^2} \frac{\sin\sqrt{x}}{\sqrt{x}} dx$

Solution:

Let u = $\sqrt{x}$

$$du = \frac{1}{2\sqrt{x}}dx$$

$$\frac{dx}{2\sqrt{x}} = 2du$$

U.L = $\sqrt{\pi^2}$= π ,   L.L = $\sqrt{\frac{\pi^2}{4}} = \frac{\pi}{2}$

$$\int_{\pi/2}^{\pi} \sin u \,.\, 2du = \;\; 2\int_{\pi/2}^{\pi} \sin u \,.\, du$$

= $2[\cos u]_{\pi/2}^{\pi}$ = -2 [cos π – cos π/2]

= - 2(-1 -0) = 2

5.7 Numerical Integration:

The numerical integration methods are approximate rules for evaluating definite integral. It used when we cannot compute the value of an integral exactly and specially useful for approximately integral of functions that are available only in graphical or tabular form.

In the present study, we will deal with three methods:

1. Trapezoidal method
2. Mid point method
3. Simpson's rule method.

5.7.1 Trapezoidal Method:

If we have the function *y = f(x)* and we want to estimate $\int_a^b f(x)dx$

If we have *n* divisions

$h = \frac{b - a}{n}$

A = total area under the curve *y*

= *f(x)* from *x* = *a* to *x* = *b*

A ≈ $a_1 + a_2 + a_3 + \ldots.. a_n = \sum_{i=1}^{n} a_i$

$a_1 = \left(\frac{y_o+y_1}{2}\right) h,\ a_2 = \left(\frac{y_1+y_2}{2}\right) h,\ a_3 = \left(\frac{y_2+y_3}{2}\right) h,\ a_n = \left(\frac{y_{n-1}+y_n}{2}\right) h$

A = $\left(\frac{y_o+y_1}{2}\right) h + \left(\frac{y_1+y_2}{2}\right) h + \left(\frac{y_2+y_3}{2}\right) h + \ldots. \left(\frac{y_{n-1}+y_n}{2}\right) h$

A ≈ h $[\frac{y_0}{2} + \frac{y_1}{2} + \frac{y_1}{2} + \frac{y_2}{2} + \frac{y_2}{2} + \frac{y_3}{2} + \frac{y_3}{2} + \ldots\ldots \frac{y_{n-1}}{2} + \frac{y_{n-1}}{2} + \frac{y_n}{2}]$

A ≈ h $[\frac{y_0}{2} + y_1 + y_2 + y_3 + \ldots\ldots y_{n-1} + \frac{y_n}{2}]$

$A = \frac{h}{2}[y_0 + 2y_1 + 2y_2 + \ldots\ldots.. + 2y_{n-1} + y_n]$

Where,

$y_0 \rightarrow y_n$: values of the function *f(x)* at $X_0 \rightarrow X_n$ y_0

= *f(a)* , y_n = *f(b)*

h = width of Trapezoid = (b – a)/n *n*

= number of divisions

Example 9: use Trapezoidal rule to approximate $\int_1^6 (x^3 + 3)dx$; n = 6, then compare the result with the exact value and find the percentage of error.

Solution:

$h = \frac{6 - 1}{6} = \frac{5}{6}$

$$A = \frac{h}{2}[y_0 + 2y_1 + 2y_2 + 2y_3 + 2y_4 + 2y_5 + y_6]$$

x	$y = f(x) = x^3 + 3$
$x_0 = a = 1$	$f(a) = y0 = 4$
$x_1 = 1 + (5/6) = 11/6$	$y_1 = 9.16$
$x_2 = 16/6$	$y_2 = 21.96$
$x_3 = 21/6$	$y_3 = 45.87$
$x_4 = 26/6$	$y_4 = 84.37$
$x_5 = 31/6$	$y_5 = 140.92$
$x_6 = 36/6 = 6$	$y_6 = 219$

A = $\frac{5}{12}$ [4 + 2(9.16) + 2(21.96) + 2(45.87) + 2(84.37) + 2(140.92) +219]

= 344.82

The exact value = $\int_1^6 (x^3 + 3)dx$

$= [\frac{x^4}{4} + 3x]_1^6 = 338.75$

The percentage of error = $\frac{344.82-338.75}{338.75} . 100$= 1.8 %

Note: the error estimated from Trapezoidal method = E_T

$$E_T = T - \int_a^b f(x)dx$$

T – approximated value of integral by Trapezoids

$$|E_T| \leq \frac{b-a}{12} h^2 D$$

D- Upper bound for the value of $|f(x)|$ on [a, b]

Example 10: Find the upper bound error estimate from using the Trapezoidal method with n = 10 for the integral $\int_0^1 x \, \sin x$

Solution: a = 0, b = 1, n = 10, h = (1 –

0)/10 = 1/10 *f(x) = x sinx f(x) = x cos x +*

sinx

f(x) = x (-sinx) + cos x + cos x = 2 cosx – x sin x
D = 2

$$|E_T| \le \frac{b-a}{12} h^2 D$$

$$|E_T| \le \frac{1}{12}\left(\frac{1}{10}\right)^2 (2)$$

$$|E_T| \le \frac{1}{600}$$

If n = 0 $|E_T| \le \frac{1}{12}\left(\frac{1}{100}\right)^2 (2)$ $|E_T| \le \frac{1}{60000}$

5.7.2 Mid-Point Method

In this method, the area under the curve is divided into a number of rectangles. The curve intersects each rectangle at the mid-point of the top side.

$$A \approx M = \sum_{i=1}^{n} f(ck) . h$$

n = number of rectangles

C_k : X- coordinate for the midpoint.

The error estimate for midpoint method is:

$$|E_M| \le \frac{b-a}{24} h^2 D$$

Example 11: Estimate $\int_1^2 x^2 dx$ with n = 4 by Midpoint method.

Solution: a = 1, b = 2, n = 4 h
= (2-1)/4 = ¼

C_k	f (C_k)
C1 = 1 + (1/4)/2 = 9/8	f(C1) = $(9/8)^2$ = 81/64
C2 = 9/8 + ¼ = 11/8	f(C2) = $(11/8)^2$ = 121/64
C3 = 11/8 + ¼ = 13/8	f(C3) = 169/64
C4 = 13/8 + ¼ = 15/8	f(C4) = 225/64

A ≈ ¼ [81/64 + 121/64 + 169/64 + 225/64] = 149/64 = 2.328125 *f*

(x) = x^2, *f(x)* = 2*x*, *f(x)* = 2, *D* = 2

$$|E_M| \leq \frac{2-1}{24}\left(\frac{1}{4}\right)^2 (2)$$

$$|E_M| \leq \frac{1}{192}$$

$$|E_M| \leq 0.005208$$

5.7.3 Simpson's Rule

Simpson's rule is based on approximately curves with parabolas instead of line segments. Each three points are connected with a parabola. The general equation of parabola is $y = Ax^2 + Bx + C$

da = y dx = $\int_{-h}^{h} y dx$

$$Ar = \int_{-h}^{h} (Ax^2 + Bx + C)dx$$

$$Ar = [\frac{Ax^3}{3} + \frac{Bx^2}{2} + Cx]_{-h}^{h}$$

$$Ar = \frac{Ah^3}{3} + \frac{Bh^2}{2} + Ch - (\frac{-Ah^3}{3} + \frac{Bh^2}{2} - Ch)$$

$$= \frac{2Ah^3}{3} + 2Ch$$

$$Ar = \frac{1}{3}(2Ah^3 + 6Ch) \ = \frac{h}{3}(2Ah^2 + 6C)\text{= Exact Area}$$

Since the curve passes through the points (-h, y_0), (0, y_1), (h, y_2) then:

$Y_0 = Ah^2 – Bh + C$ ……. (1)
$Y_1 = C$ … …………………….(2)

$Y_2 = Ah^2 + Bh + C$ ……..(3)

$Y_0 – y_1 = Ah^2 – Bh$

$Y_2 – y_1 = Ah^2 + Bh$

$Y_0 – y_1 + y_2 – y_1 = 2Ah^2$ $\quad Ar = \frac{h}{3}[y_0 + y_2 - 2y_1 + 6C]$

$$Ar = \frac{h}{3}[y_0 + y_2 - 2y_1 + 6C] \ = \frac{h}{3}[y_0 + 4y_1 + y_2]$$

Now: $Ar1 = \frac{h}{3}[y_0 + 4y_1 + y_2]$

$$Ar2 = \frac{h}{3}[y_2 + 4y_3 + y_4]$$

Total area = Ar1 + Ar2 $= \frac{h}{3}[y_0 + 4y_1 + y_2 + y_2 + 4y_3 + y_4]$

$$= \frac{h}{3}[y_0 + 4y_1 + 2y_2 + 4y_3 + y_4]$$

$$A = S = \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots \ldots. +2y_{n-2} + 4y_{n-1} + y_n)$$

Example 12: Estimate $\int_1^2 x^2 dx$ with n = 4 by Simpson’s rule.

Solution: a = 1, b = 2, n = 4 h

= (2 – 1)/4 = ¼

$$A = S = \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4)$$

x	*y*
$x_0 = 1$	$y_0 = 1$

$x_1 = 5/4$	$y_1 = 25/16$
$x_2 = 6/4$	$y_2 = 36/16$
$x_3 = 7/4$	$y_3 = 49/16$
$x_4 = 2$	$y_4 = 4$

$$A = \frac{1}{4 * 3}\left[1 + 4\left(\frac{25}{16}\right) + 2\left(\frac{36}{16}\right) + 4\left(\frac{49}{16}\right) + 4\right]$$

= 7/3 = 2.333333

The error estimate for Simpson's rule is :

$$|E_S| \leq \frac{b-a}{180} h^4 D$$

D – Upper bound for the value of $\left|f^{(4)}\right|$ on [a, b]

□ When the power of $f(x) \leq 3$ D = 0 error becomes Zero

S gives exact value for the integrals of the functions of third degree or less

For the above example: $f(x) = x^2$, $f(x) = 2x$, $f(x) = 2$, $f(x) = 0$, $f(x) = 0$ $D = 0$

The error = 0

Example 13: Determine n that will guarantee an accuracy of at least 10^{-7} for using:

1. Trapezoidal rule
2. Simpson's rule

To approximate $\int_2^4 x^4 dx$

Solution:

1. By Trapezoidal rule:

$$|E_T| \leq \frac{b-a}{12} h^2 D$$

a = 2, b = 4, h = (b – a)/n = 2/n

$f(x) = x^4$ $f(x) = 4x^3$ $f(x) = 12x^2$

at x = 4 $\quad f(4) = (4)^2 * 12 = 192 \quad D = 192$

$$|E_T| \leq \frac{4-2}{12}\left(\frac{2}{n}\right)^2 * 192 \qquad |E_T| \leq \frac{128}{n^2}$$

$\frac{128}{n^2} \leq 10^{-7}$ $\quad n^2 \geq 128 * 10^7$ $\quad n \geq 35777.08$ $\quad n \geq 35778$

2. By Simpson's rule

$$|E_S| \leq \frac{b-a}{180} h^4 D$$

$f(x) = 12\, x^2\, f(x) = 24\, x$

$f(x) = 24 \quad D = 24$

$|E_S| \leq \frac{2}{180}\left(\frac{2}{n}\right)^4 * 24$ $\qquad |E_S| \leq \frac{64}{15n^4}$

$\frac{64}{15n^4} \leq 10^{-7}$ $\quad n \geq 80.82 \quad n = 82$

Example 14: The table below shows the velocity of submarine with the travelling time. Use Simpson's rule to estimate the distance travelled during the 10 hours period.

t (hr)	*v (mph)*
0	12
1	14
2	17
3	21
4	22
5	21
6	15
7	11
8	11
9	14
10	17

Solution:

$v = \frac{ds}{dt}$ $\qquad$ *ds = v. dt*

$$s = \int_0^{10} v(t).dt$$

h = 1, n = 10

S = 1/3 [12 + 4(14) + 2 (17) + 4(21) + 2(22) + 4(21) + 2(15) + 4(11) + 2(11) + 4(14) + 17] *S* = 161 mile

5.8 Indefinite Integrals:

If the function *f(x)* is a derivative, then the set of all antiderivatives of *f* is called the definite integral of *f*.

The form of indefinite integral is $\int f(x)dx$

The value of this integral is F (x) + C

F(x): Antiderivative

C: Constant of integration (arbitrary constant)

$\int f(x)dx$ = F (x) + C

5.9 Integration Formulas:

If *u* = *f(x)*

1. $\int u^n \, du = \frac{u^{n+1}}{n+1} + c$
2. $\int \sin u \, du = -\cos u + c$
3. $\int \cos u \, du = \sin u + c$
4. $\int sec^2 u \, du = \tan u + c$
5. $\int \sec u \tan u \, du = \sec u + c$
6. $\int csc^2 u \, du = -\cot u + c$
7. $\int \csc u \cot u \; du = -\csc u + c$

Example 15: $\int(3x^2 - 2x + 5)\, dx$

$$= 3\int x^2\, dx - 2\int x\, dx + 5\int dx$$

$= 3\ (x^3/3) - 2\ (x^2/2) + 5x + C$

$= x^3 - x^2 + 5x + c$

Example 16: $\int \cos^2 x\, dx$

$= \int \frac{1+\cos 2x}{2} dx = \int \frac{1}{2} dx + \int \frac{1}{2} \cos 2x\, dx$

$$\int \frac{1}{2}\cos 2x\, dx = \frac{1}{2}\int \cos 2x\, dx = \frac{1}{2}\cdot\frac{1}{2}\int \cos 2x \cdot 2dx$$

$$= \frac{1}{4}\int \cos 2x \cdot 2dx = \frac{1}{4}\sin 2x$$

$$\int \cos^2 x\, dx = \frac{x}{2} + \frac{1}{4}\sin 2x + C$$

Example 17: Solve the following differential equations:

1. $dy/dx = x^2 + 1$
2. $dy/dx = \sqrt{xy},\quad x > 0, y > 0$
3. $x^3(dy/dx) = -2, x > 0$ **Solution**:

1. $dy/dx = x^2 + 1 \qquad dy = (x^2 + 1)\, dx$

$$\int dy = \int(x^2 + 1)\, dx \qquad y = x^3/3 + x + C$$

2. $dy/dx = \sqrt{xy} \qquad dy/dx = (xy)^{1/2} \qquad dy = (xy)^{1/2}dx$

$dy = x^{1/2}\, y^{1/2}\, dx \qquad y^{-1/2}\, dy = x^{1/2}\, dx$

$$\int y^{-1/2} dy = \int x^{1/2} dx$$

$$\frac{y^{1/2}}{\frac{1}{2}} = \frac{x^{3/2}}{\frac{3}{2}} + C \qquad 2y^{1/2} = \frac{2}{3}x^{3/2} + C \qquad 2\sqrt{y} = \frac{2}{3}\sqrt{x^3} + C$$

3. $x^3(dy/dx) = -2 \qquad dy = -2\, x^{-3}\, dx$

$\int dy = -2\int x^{-3}dx \qquad y = -2\left(\frac{x^{-2}}{-2}\right) + C$

$$y = \frac{1}{x^2} + C$$

5.10 Integration by Substitution:

If we have the integral $\int f\big(g(x)\big).\,g(x)dx$, substitute *u = g (x), du = g(x)* to obtain $\int f(u)du$.

Example 18: Evaluate $\int \cos\sqrt{x}\,\frac{dx}{\sqrt{x}}$

Solution: let $u = \sqrt{x} \qquad \frac{du}{dx} = \frac{1}{2}x^{-1/2} \qquad du = \frac{1}{2}x^{-1/2}dx$

$x^{-1/2}$ dx = 2 du

$$\int \cos\sqrt{x}\,\frac{dx}{\sqrt{x}} = \int \cos x^{1/2}\cdot x^{1/2}dx = \int \cos u \cdot 2\,du$$

$= 2\int \cos u \cdot du = 2\sin u + c = 2\sin\sqrt{x} + c$

Example 19: $\int \frac{x\cos\sqrt{3x^2-6}}{\sqrt{3x^2-6}}dx$

Let $u = \sqrt{3x^2 - 6} \qquad \frac{du}{dx} = \frac{1}{2\sqrt{3x^2-6}}\cdot 6x$

$du = \frac{3x\,dx}{\sqrt{3x^2-6}} \qquad \frac{x\,dx}{\sqrt{3x^2-6}} = \frac{du}{3}$

$$\int \cos u \cdot \frac{du}{3} = \frac{1}{3}\int \cos u\,du$$

$$= \frac{1}{3}\sin\sqrt{3x^2 - 6} + c$$

Example 20: Evaluate $\int \sin 3x \cos 3x\,dx$ by three different methods

Solution:

Method 1: $\int(\sin 3x)\;\cos 3x\,dx$

Let *u = sin 3x* *du = 3 cos 3x dx* *cos 3x dx = du/3*

$$\int u\frac{du}{3} = \frac{1}{3}\int u\, du = \frac{1}{3}\cdot\frac{u^2}{2} + c = \frac{1}{6}sin^2 3x + c1$$

Method 2: $\int(\cos 3x)\ \sin 3x\, dx$

Let $u = \cos 3x$ $\quad du = -3 \sin 3x\, dx$ $\quad \sin 3x\, dx = -du/3$

$$\int u\frac{-du}{3} = -\frac{1}{3}\int u\, du = -\frac{1}{3}\cdot\frac{u^2}{2} + c = \frac{-1}{6}cos^2 3x + c2$$

Method 3: $\int \sin 3x\ \cos 3x\, dx\ = \frac{1}{2}\int 2\sin 3x\ \cos 3x\, dx$

$$= \frac{1}{2}\int \sin 6x\ dx = \frac{1}{2}\cdot\frac{1}{6}\int \sin 6x\cdot 6\, dx = \frac{1}{12}(-\cos 6x) + c3$$

Solving the Initial Value Problems:

Given $dy/dx = f(x)$ $\quad dy = f(x)\, dx$ $\quad \int dy = \int f(x)\, dx\ y$

$= F(x) + C$ $\quad$ *general solution*

Use initial condition to find C

Example 21: Find the equation of a curve whose slope is $\frac{x}{\sqrt{x^2+3}}$ and passes through the point (1, 1).

Solution: Slope $=\frac{dy}{dx} = \frac{x}{\sqrt{x^2+3}}$ $\quad dy\,(x^2 + 3)^{1/2} = x\, dx$

$dy = (x^2 + 3)^{-1/2}\, x\, dx$ $\quad \int dy = \int (x^2 + 3)^{-1/2}\, x\, dx$

$y = \int (x^2 + 3)^{-1/2}\,(2/2)\, x\, dx\ y = 1/2\int (x^2 + 3)^{-1/2}$

$(1/2)\, x\, dx\ y = ½ *2\,(x^2 + 3)^{½} + C$

$y = \sqrt{x^2+3}+C$ *(general solution)*

At point (1,1) $\quad 1 = \sqrt{1+3} + C$ $\quad c = -1$

$y = \sqrt{x^2+3}\ -1$ *(particular solution)*

Example 22: Solve $2y\,(dy/dx) = 5x - \sin x$ $\quad x = 0, y = 0$

Solution: $2y\, dy = (5x - \sin x)\, dx$ $\quad 2y\, dy = 5x\, dx - \sin x\, dx$

$\int 2y\, dy = \int 5x\, dx - \int \sin x\, dx$ $\quad 2\frac{y^2}{2} = 5\frac{x^2}{2} + \cos x + C$

$y^2 = (5/2)\, x^2 + \cos x + C$

at point (0, 0) $\quad 0 = 0 + 1 + c \quad c = -1\ y^2$

$= (5/2)\, x^2 + \cos x - 1$

Example 23: Solve the differential equation $dy/dx = 3x^2 - 2x - 1$, the initial conditions (y = 10 at x = 1).

Solution:

$dy/dx = 3x^2 - 2x - 1 \qquad \int dy = \int (3x^2 - 2x - 1)\, dx\ y$

$= x^3 - x^2 - x + c$ *(general solution)*

at $x = 1, y = 10 \qquad 10 = (1)^2\ (1)^2 - 1 + c \qquad c = 11\ y$

$= x^3 - x^2 - x + 11$ *(particular solution)*