In this lecture we define separated and disconnected also we study the proprieties and results of the separated and connected of spaces . Let (X , T) be a topological space . Two non-empty subsets A and B of X are said to be T- separated if and only if the intersection of A and cl(B) be empty and also the intersection of B and cl(A) be empty . Note that A and B are separated then A and B are disjoint . Thus A and B are separated if and only if A and B are disjoint and neither of them contains limit points of the other .Let (Y , T_Y) be subspace of a topological space (X , T) and let A and B be two subsets of Y , A and B are T – separated if and only if they are T_Y – separated . If A and B are separated subsets of a space X and C subset of A and D subset of B , then C and D are also separated . Two closed (open) subsets A , B of a topological space are separated if and only if they are disjoint . Two disjoint sets A ,B are separated in a topological space (X , T) if and only if they are both open and closed in the subspace of the union of A and B . Let (X , T) be topological space . A subset A of X is said to be T – disconnected if and only if it is the union of two non-empty T-separated . A is said to be connected if and only if it is not disconnected . The empty set is trivially connected . Also every singleton set in a space is connected since it cannot be expressed as a union of two non-empty separated sets .

Definition 95. Let (X, τ) be a top. space. A s
A of X is said to be τ-disconnected iff it is
union of two non-empty τ-separated, that i
$\exists$ two non-empty sets C and $D \ni C \cap \bar{D} =$
$\bar{C} \cap D = \phi$ and $A = C \cup D$.

A is said to be connected iff it is not di

Remark. The empty set is trivially connecte
Also every singleton set in a space is connec
since it cannot be expressed as a union of
non-empty separated sets.

Ex. Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b, c\}\}$
Show that X is disconnected.

Solution. Since $\{a\}, \{b, c\}$ be open and clo
sets in X.

So $\overline{\{a\}} = \{a\}$ and $\overline{\{b, c\}} = \{b, c\}$

And $\{a\} \cap \overline{\{b, c\}} = \phi$ and $\overline{\{a\}} \cap \{b, c\} = \phi$

$X = \{a\} \cup \{b, c\}$. Thus X is disconnect

Definition 96. Two points a and b of a top. s
X are said to be connected iff they are cont
in a connected subset of X.

$$cl_E(A) = cl_X(A) \cap (A \cup B)$$
$$= [cl_X(A) \cap A] \cup [cl_X(A) \cap B]$$
$$= A \cup \phi = A$$

Since $A \subset cl_X(A) \Rightarrow A \cap cl_X(A) = A$

and $cl_X(A) \cap B = \phi$.

Thus $cl_E(A) = A \Rightarrow A$ is closed in t

subspace $A \cup B$ Similarly B is close

subspace $A \cup B$. And $A \cap B = \phi \Rightarrow$

$(A \cup B) - A = B$ and $(A \cup B) - B = A$

Thus B and A are opens in the subspace

Conversely, let the disjoint sets A and B b

open and closed in $A \cup B$. T.P. A and B are se

in X. Since A is closed in $E \Rightarrow$

$$A = cl_E(A) = [cl_X(A) \cap E] = cl_X(A) \cap [A \cup$$
$$= (cl_X(A) \cap A) \cup (cl_X(A) \cap B)$$
$$A = A \cup [cl_X(A) \cap B] \text{ Since } A \subset cl_X$$

Thus $(cl_X(A) \cap B) \subset A$

$$\Rightarrow [(cl_X(A) \cap B) \cap B] \subset A \cap B = \phi$$
$$\Rightarrow cl_X(A) \cap B \subset \phi \Rightarrow cl_X(A) \cap B =$$

Theorem 93. Two closed (open) subsets A, B of a space are separated iff they are disjoint.

proof. $\Rightarrow$ Since any two separated sets are

$\Leftarrow$ let A, B be two closed disjoint.

$\Rightarrow A = \bar{A}$ and $B = \bar{B}$ and $A \cap B = \phi$

Thus $\phi = A \cap B = A \cap \bar{B}$ and $\phi = A \cap B = \bar{A} \cap B$

Hence A and B are separated

Now let A and B are both open and disjoint

$\Rightarrow A^c$ and B^c are both closed s

$\overline{A^c} = A^c$ and $\overline{B^c} = B^c$ and $A \cap B =$

Thus $A \subset B^c$ and $B \subset A^c$

$\Rightarrow \bar{A} \subset \overline{B^c} = B^c$ and $\bar{B} \subset \overline{A^c} = A^c$

Hence $\bar{A} \cap B = \phi$ and $\bar{B} \cap A = \phi$

Therefore A and B are separated.

Theorem 94. Two disjoint sets A and B are separated in a top-space $(X, \mathcal{T})$ iff they are both open and closed in the subspace $A \cup B$.

proof. $\Rightarrow$ let A and B be separated in X

$\Rightarrow A \cap cl_X(B) = \phi$ and $cl_X(A) \cap B = \phi$

Theorem 91. Let (Y, τ_Y) be a subspace of a space (X, τ) and let A and B be two sub
Then A, B are τ- separated iff they are τ_Y

proof. $cl_Y(A) = cl_X(A) \cap Y$ and $cl_Y(B) = cl_X($

Now $(cl_Y(A) \cap B) \cup (A \cap cl_Y(B))$
$= (cl_X(A) \cap B \cap Y) \cup (A \cap cl_X(B) \cap Y)$
$= (cl_X(A) \cap B) \cup (A \cap cl_X(B))$
because A, B are subsets of Y
Thus $A \cap Y = A$ & $B \cap Y = B$
Hence $(cl_Y(A) \cap B) \cup (A \cap cl_Y(B)) = \phi$
$= (cl_X(A) \cap B) \cup (A \cap cl_X(B))$.

Theorem 92. If A and B are separated subs a space X and $C \subset A$ and $D \subset B$, then C and also separated

proof. We are given that $A \cap \bar{B} = \phi$ and $\bar{A} \cap B$
Also $C \subset A \Rightarrow \bar{C} \subset \bar{A}$ and $D \subset B \Rightarrow \bar{D} \subset \bar{B}$
Thus $C \cap \bar{D} = \phi$ and $\bar{C} \cap D = \phi$. Hence C
are separated.

Definition 90. Let (X,τ) be a top. space. non-empty subsets A and B of X are said t
τ-separated iff $A\cap\bar{B}=\phi$ and $\bar{A}\cap B=$
i.e. $(A\cap\bar{B})\cup(\bar{A}\cap B)=\phi$.

Note ① A and B are separated then

Since $A\cap\bar{B}=\phi$ and $B\subset\bar{B}\Rightarrow A\cap B=$

② Since $A\cap\bar{B}=\phi\Rightarrow A\cap(B\cup D(B))=\phi$

$(A\cap B)\cup(A\cap D(B))=\phi$

Since $A\cap B=\phi$

$\Rightarrow A\cap D(B)=\phi$ Also $D(A)$

Thus A and B are separated iff A and B
disjoint and neither of them contains lim
of the other.

③ If two disjoint sets are not necessari
separated.

Ex let $A=(-\infty,0)$ and $B=[0,\infty)$ be
of usual top. $\mathbb{R}$. Thus $A\cap B=\phi$. b
$\bar{A}=[-\infty,0]$ and $\bar{A}\cap B=[-\infty,0]\cap[0,$

Thus A and B are not separated