BOOLEAN ALGEBRA

2.1 Introduction

Binary logic deals with variables that have two discrete values: 1 for *TRUE* and 0 for *FALSE*. A simple switching circuit containing active elements such as a diode and transistor can demonstrate the binary logic, which can either be ON (switch closed) or OFF (switch open). Electrical signals such as voltage and current exist in the digital system in either one of the two recognized values, except during transition.

The switching functions can be expressed with Boolean equations. Complex Boolean equations can be simplified by a new kind of algebra, which is popularly called Switching Algebra or Boolean Algebra, invented by the mathematician George Boole in 1854. Boolean Algebra deals with the rules by which logical operations are carried out.

2.2 BASIC DEFINITIONS

Boolean algebra, like any other deductive mathematical system, may be defined with a set of elements, a set of operators, and a number of assumptions and postulates. A set of elements means any collection of objects having common properties. If S denotes a set, and X and Y are certain objects, then X ∈ S denotes X is an object of set S, whereas Y ∉ S denotes Y is not the object of set S. A binary operator defined on a set S of elements is a rule that assigns to each pair of elements from S a unique element from S. As an example, consider this relation X*Y = Z. This implies that * is a binary operator if it specifies a rule for finding Z from the objects (X, Y) and also if all X, Y, and Z are of the same set S. On the other hand, * can not be binary operator if X and Y are of set S and Z is not from the same set S.

The postulates of a mathematical system are based on the basic assumptions, which make possible to deduce the rules, theorems, and properties of the system. Various algebraic structures are formulated on the basis of the most common postulates, which are described as follows:

1. Closer: A set is closed with respect to a binary operator if, for every pair of elements of S, the binary operator specifies a rule for obtaining a unique element of S. For example, the set of natural numbers N = {1, 2, 3, 4, ...} is said to be closed with respect to the binary operator plus (+) by the rules of arithmetic addition, since for any X,Y ∈ N we obtain a unique element Z ∈ N by the operation X + Y = Z. However, note that the set of natural numbers is not closed with respect to the binary operator minus (–) by the rules of arithmetic subtraction because for 1 – 2 = –1, where –1 is not of the set of naturals numbers.

2. Associative Law: A binary operator * on a set S is said to be associated whenever (A*B)*C = A*(B*C) for all A,B,C ∈ S.

3. Commutative Law: A binary operator * on a set S is said to be commutative Whenever A*B = B*A for all A,B ∈ S.

4. Identity Element: A set S is to have an identity element with respect to a binary operation * on S, if there exists an element E ∈ S with the property E*A = A*E = A.

Example: The element 0 is an identity element with respect to the binary operator + on the set of integers I = {.... –4, –3, –2, –1, 0, 1, 2, 3, 4, ...} as
A + 0 = 0 + A = A.
Similarly, the element 1 is the identity element with respect to the binary operator × as A × 1 = 1 × A = A.

5. Inverse: If a set S has the identity element E with respect to a binary operator *, there exists an element B ∈ S, which is called the inverse, for every A ∈ S, such that A*B = E.

Example: In the set of integers I with E = 0, the inverse of an element A is (-A) since A + (–A) = 0.
6. Distributive Law: If * and (.) are two binary operators on a set S, * is said to be distributive over (.), whenever
A*(B.C) = (A*B).(A*C).
If summarized, for the field of real numbers, the operators and postulates have the following meanings:
The binary operator + defines addition.
The additive identity is 0.
The additive inverse defines subtraction.
The binary operator (.) defines multiplication.
The multiplication identity is 1.
The multiplication inverse of A is 1/A, defines division *i.e.,* A. 1/A = 1.
The only distributive law applicable is that of (.) over +
A . (B + C) = (A . B) + (A . C)

2.3 DEFINITION OF BOOLEAN ALGEBRA

In 1854 George Boole introduced a systematic approach of logic and developed an algebraic system to treat the logic functions, which is now called Boolean algebra. In 1938 C.E. Shannon developed a two-valued Boolean algebra called Switching algebra, and demonstrated that the properties of two-valued or bistable electrical switching circuits can be represented by this algebra. The postulates formulated by E.V. Huntington in 1904 are employed for the formal definition of Boolean algebra. However, Huntington postulates are not unique for defining Boolean algebra and other postulates are also used. The following Huntington postulates are satisfied for the definition of Boolean algebra on a set of elements S together with two binary operators (+) and (.).
1. (*a*) Closer with respect to the operator (+).
(*b*) Closer with respect to the operator (.).
2. (*a*) An identity element with respect to + is designated by 0 *i.e.,* A + 0 = 0 + A = A.

(*b*) An identity element with respect to . is designated by 1 *i.e.*, A.1 = 1. A = A.
3. (*a*) Commutative with respect to (+), *i.e.*, A + B = B + A.
(*b*) Commutative with respect to (.), *i.e.*, A.B = B.A.
4. (*a*) (.) is distributive over (+), *i.e.*, A . (B+C) = (A . B) + (A . C).
(*b*) (+) is distributive over (.), *i.e.*, A + (B .C) = (A + B) . (A + C).
5. For every element A ∈ S, there exists an element A′ ∈ S (called the complement of A) such that A + A′ = 1 and A . A′ = 0.
6. There exists at least two elements A,B ∈ S, such that A is not equal to B.

Comparing Boolean algebra with arithmetic and ordinary algebra (the field of real numbers), the following differences are observed:

1. Huntington postulates do not include the associate law. However, Boolean algebra follows the law and can be derived from the other postulates for both operations.
2. The distributive law of (+) over (.) *i.e.*, A+ (B.C) = (A+B) . (A+C) is valid for Boolean algebra, but not for ordinary algebra.
3. Boolean algebra does not have additive or multiplicative inverses, so there are no subtraction or division operations.
4. Postulate 5 defines an operator called Complement, which is not available in ordinary algebra.
5. Ordinary algebra deals with real numbers, which consist of an infinite set of elements. Boolean algebra deals with the as yet undefined set of elements S, but in the two valued Boolean algebra, the set S consists of only two elements: 0 and 1.

Boolean algebra is very much similar to ordinary algebra in some respects. The symbols (+) and (.) are chosen intentionally to facilitate Boolean algebraic manipulations by persons already familiar to ordinary algebra. Although one can use some knowledge from ordinary algebra to deal with Boolean algebra, beginners must be careful not to substitute the rules of ordinary algebra where they are not applicable.

2.4 TWO-VALUED BOOLEAN ALGEBRA

Two-valued Boolean algebra is defined on a set of only two elements, S = {0,1}, with rules for two binary operators (+) and (.) and inversion or complement as shown in the following operator tables at Figures 2.1, 2.2, and 2.3 respectively.

A	B	A + B
0	0	0
0	1	1
1	0	1
1	1	1

Figure 2-1

A	B	A.B
0	0	0
0	1	0
1	0	0
1	1	1

Figure 2-2

A	A′
0	1
1	0

Figure 2-3

The rule for the complement operator is for verification of postulate 5.

These rules are exactly the same for as the logical OR, AND, and NOT operations, respectively. It can be shown that the Huntington postulates are applicable for the set S = {0,1} and the two binary operators defined above.

1. Closure is obviously valid, as form the table it is observed that the result of each operation is either 0 or 1 and 0,1 ∈ S.
2. From the tables, we can see that:

(*i*) 0 + 0 = 0   0 + 1 = 1   1+ 0 = 1

(*ii*) 1 . 1 = 1   0 . 1 = 0   0.0 = 0

which verifies the two identity elements 0 for (+) and 1 for (.) as defined by postulate 2.

3. The commutative laws are confirmed by the symmetry of binary operator tables.
4. The distributive laws of (.) over (+) *i.e.*, A . (B+C) = (A . B) + (A . C), and (+) over (.) *i.e.*, A + (B . C) = (A+B) . (A+C) can be shown to be applicable with the help of the truth tables considering all the possible values of A, B, and C as under.

From the complement table it can be observed that:

(a) Operator (.) over (+)

A	B	C	B+C	A.(B+C)	A.B	A.C	(A.B)+(A.C)
0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	0	0	0	0
1	0	0	0	0	0	0	0
1	0	1	1	1	0	1	1
1	1	0	1	1	1	0	1
1	1	1	1	1	1	1	1

Figure 2-4

(b) Operator (+) over (.)

A	B	C	B.C	A+(B.C)	A+B	A+C	(A+B).(A+C)
0	0	0	0	0	0	0	0
0	0	1	0	0	0	1	0
0	1	0	0	0	1	0	0
0	1	1	1	1	1	1	1
1	0	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1

Figure 2-5

(*c*) A + A′ = 1, since 0 + 0' = 1 and 1 + 1' = 1.

(*d*) A . A′ = 0, since 0 . 0' = 0 and 1 . 1' = 0.

These confirm postulate 5.

5. Postulate 6 also satisfies two-valued Boolean algebra that has two distinct elements 0 and 1 where 0 is not equal to 1.

2.5 BASIC PROPERTIES AND THEOREMS OF BOOLEAN ALGEBRA

2.5.1 DeMorgan's Theorem

Two theorems that were proposed by DeMorgan play important parts in Boolean algebra.

The first theorem states that the complement of a product is equal to the sum of the complements. That is, if the variables are A and B, then:

$$(A.B)' = A' + B'$$

The second theorem states that the complement of a sum is equal to the product of the complements. In equation form, this can be expressed as:

$$(A + B)' = A' . B'$$

The complements of Boolean logic function or a logic expression may be simplified or expanded by the following steps of DeMorgan's theorem.

(*a*) Replace the operator (+) with (.) and (.) with (+) given in the expression.

(*b*) Complement each of the terms or variables in the expression.

DeMorgan's theorems are applicable to any number of variables. For three variables A, B, and C, the equations are:

$$(A.B.C)' = A' + B' + C' \text{ and}$$
$$(A + B + C)' = A'.B'.C'$$

The following is the complete list of postulates and theorems useful for two-valued Boolean algebra.

Postulate 2	(*a*) $A + 0 = A$	(*b*) $A.1 = A$
Postulate 5	(*a*) $A + A' = 1$	(*b*) $A.A' = 0$
Theorem 1	(*a*) $A + A = A$	(*b*) $A.A = A$
Theorem 2	(*a*) $A + 1 = 1$	(*b*) $A.0 = 0$
Theorem 3, Involution	$(A')' = A$	
Theorem 3, Commutative	(*a*) $A + B = B + A$	(*b*) $A.B = B.A$
Theorem 4, Associative	(*a*) $A + (B + C) = (A + B) + C$	(*b*) $A.(B.C) = (A.B).C$
Theorem 4, Distributive	(*a*) $A(B + C) = A.B + A.C$	(*b*) $A + B.C = (A + B).(A + C)$
Theorem 5, DeMorgan	(*a*) $(A + B)' = A'.B'$	(*b*) $(A.B)' = A' + B'$
Theorem 6, Absorption	(*a*) $A + A.B = A$	(*b*) $A.(A + B) = A$

Figure 2-6