LAMINAR BOUNDARY LAYER ON A FLAT PLATE

Newton's second law of motion, $F_x = d(mV)/dt$

The rate of mass flow rate $\dot{m} = \rho u A$

$$d\dot{m} = \rho\, u\, dA$$

$$\dot{m} = \int \rho u dA$$

Continuity equation for the boundary layer $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

The momentum equation of the laminar boundary layer with constant properties

$$\rho \left(u\, \partial u/\partial x + v\, \partial u/\partial y\right) = \mu \left(\partial^2 u/\partial y^2\right) - \left(\partial p/\partial x\right)$$

The velocity profiles at various x positions are similar; that is, they have the same functional dependence on the y coordinate. There are four conditions to satisfy. The simplest function that we can choose to satisfy these conditions is a polynomial with four arbitrary constants.

$$u = C1 + C2y + C3\, y^2 + C4y^3$$

To obtain an expression for the boundary-layer thickness. For our approximate analysis the conditions that the velocity function must satisfy:

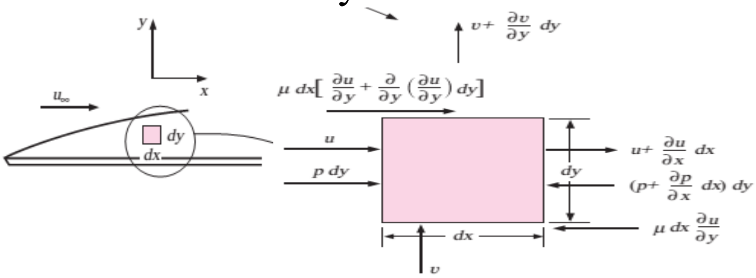
$u = 0$ at $y = 0$ …………………. [a]

$u = u_\infty$ at $y = \delta$ ………………. [b]

$\partial u/\partial y = 0$ at $y = \delta$ …………… [c]

For a constant-pressure condition

$\partial^2 u/\partial y^2 = 0$ at $y = 0$ …………… .[e]



Applying the four conditions (a) to (d)

$$\frac{u}{u\infty} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3 \qquad \text{Where } (\delta) \text{ boundary} - \text{layer thickness}$$

$u\infty$ the free-stream velocity outside the boundary layer

$$\frac{d}{dx}\left\{\rho u_\infty^2 \int_0^\delta \left[\frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right]\left[1 - \frac{3}{2}\frac{y}{\delta} + \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right] dy\right\} = \mu\, \frac{\partial u}{\partial y}\bigg]_{y=0} \quad = \frac{3}{2}\frac{\mu u_\infty}{\delta}$$

$$\frac{d}{dx}\left(\frac{39}{280}\rho u_\infty^2\,\delta\right) = \frac{3}{2}\frac{\mu u_\infty}{\delta}$$

Since ρ and u_∞ are constants, the variables may be separated to give

$$\delta\, d\delta = \frac{140}{13}\frac{\mu}{\rho u_\infty}\,dx = \frac{140}{13}\frac{\nu}{u_\infty}\,dx$$

and

$$\frac{\delta^2}{2} = \frac{140}{13}\frac{\nu x}{u_\infty} + \text{const}$$

At $x = 0$, $\delta = 0$, so that

$$\delta = 4.64\sqrt{\frac{\nu x}{u_\infty}}$$

This may be written in terms of the Reynolds number as

$$\frac{\delta}{x} = \frac{4.64}{\mathrm{Re}_x^{1/2}}$$

where

$$\mathrm{Re}_x = \frac{u_\infty x}{\nu}$$

Mass Flow and Boundary-Layer Thickness

EXAMPLE 5-3

Air at 27°C and 1 atm flows over a flat plate at a speed of 2 m/s. Calculate the boundary-layer thickness at distances of 20 cm and 40 cm from the leading edge of the plate. Calculate the mass flow that enters the boundary layer between $x = 20$ cm and $x = 40$ cm. The viscosity of air at 27°C is 1.85×10^{-5} kg/m · s. Assume unit depth in the z direction.

■ Solution

The density of air is calculated from

$$\rho = \frac{p}{RT} = \frac{1.0132 \times 10^5}{(287)(300)} = 1.177\ \mathrm{kg/m^3} \quad [0.073\ \mathrm{lb}_m/\mathrm{ft}^3]$$

The Reynolds number is calculated as

$$\text{At } x = 20 \text{ cm:} \qquad \mathrm{Re} = \frac{(1.177)(2.0)(0.2)}{1.85 \times 10^{-5}} = 25{,}448$$

$$\text{At } x = 40 \text{ cm:} \qquad \mathrm{Re} = \frac{(1.177)(2.0)(0.4)}{1.85 \times 10^{-5}} = 50{,}897$$

The boundary-layer thickness is calculated from Equation (5-21):

$$\text{At } x = 20 \text{ cm:} \qquad \delta = \frac{(4.64)(0.2)}{(25{,}448)^{1/2}} = 0.00582\ \mathrm{m} \quad [0.24\ \mathrm{in}]$$

$$\text{At } x = 40 \text{ cm:} \qquad \delta = \frac{(4.64)(0.4)}{(50{,}897)^{1/2}} = 0.00823\ \mathrm{m} \quad [0.4\ \mathrm{in}]$$

To calculate the mass flow that enters the boundary layer from the free stream between $x = 20$ cm and $x = 40$ cm, we simply take the difference between the mass flow in the boundary layer at these two x positions. At any x position the mass flow in the boundary layer is given by the integral

$$\int_0^\delta \rho u \, dy$$

where the velocity is given by Equation (5-19),

$$u = u_\infty \left[\frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3 \right]$$

Evaluating the integral with this velocity distribution, we have

$$\int_0^\delta \rho u_\infty \left[\frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3 \right] dy = \frac{5}{8}\rho u_\infty \delta$$

Thus the mass flow entering the boundary layer is

$$\begin{aligned} \Delta m &= \tfrac{5}{8}\rho u_\infty (\delta_{40} - \delta_{20}) \\ &= (\tfrac{5}{8})(1.177)(2.0)(0.0082 - 0.0058) \\ &= 3.531 \times 10^{-3} \text{ kg/s} \quad [7.78 \times 10^{-3} \text{ lb}_m\text{/s}] \end{aligned}$$

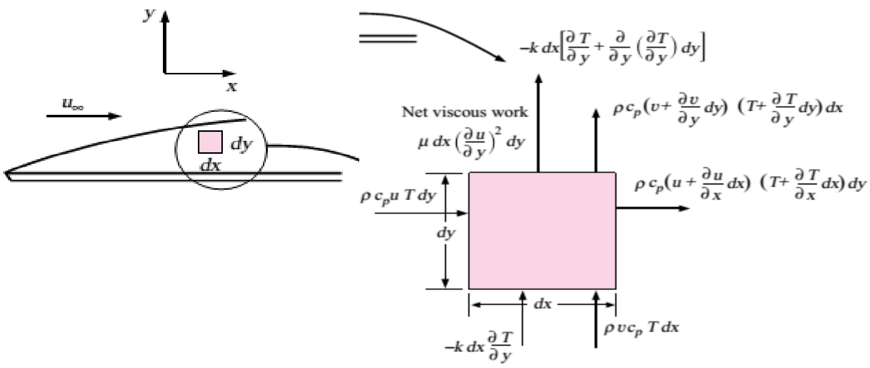
ENERGY EQUATION OF THEBOUNDARY LAYER

∂T/∂x<<∂T/∂y

the energy balance may be written : Energy convected in left face + energy convected in bottom face+ heat conducted in bottom face+ net viscous work done on element= energy convected out right face + energy convected out top face+ heat conducted out top face

where Pr is called the Prandtl number $\quad \mathrm{Pr} = \frac{\nu}{\alpha} = \frac{\mu/\rho}{k/\rho c_p} = \frac{c_p \mu}{k}$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} + \frac{\mu}{\rho c_p}\left(\frac{\partial u}{\partial y}\right)^2$$



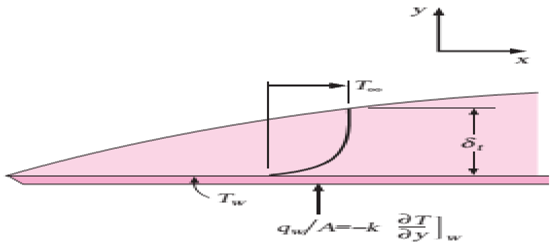
The viscous dissipation is small in comparison with the conduction term. The viscous dissipation is small for even this rather large flow velocity of 70 m/s. Thus, for low-velocity incompressible flow

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}$$

THE THERMAL BOUNDARY LAYER

For the system shown in Figure . The temperature of the wall is *Tw*, the temperature of the fluid outside the thermal boundary layer is *T*∞, and the thickness of the thermal boundary layer is designated as(*δt*). At the wall, the velocity is zero, and the heat transfer into the fluid takes place by conduction. Thus the local heat flux per unit area, *q*", is

$$\frac{q}{A} = q" = -k\frac{\partial T}{\partial y}]_{wall}$$



From Newton's law of cooling q" = h(Tw − T∞)
where h is the convection heat-transfer coefficient.
Combining these equations,

$$h = \frac{-k\frac{\partial T}{\partial y}]_{wall}}{Tw - T\infty}$$

To find the temperature gradient at the wall in order to evaluate the heat-transfer coefficient. This means that we must obtain an expression for the temperature distribution. To do this, an approach similar to that used in the momentum analysis of the boundary layer is followed. The conditions that the temperature distribution must satisfy are

$T = C1 + C2y + C3\,y^2 + C4y^3$

$T = Tw$ at $y = 0$……………. [a]

$\partial T/\partial y = 0$ at $y = \delta t$ ……….. [b]

$T = T\infty$ at $y = \delta t$ ………….[e]

$\partial^2 T/\partial y^2 = 0$ at $y = 0$………..[d]

Since the velocities must be zero at the wall

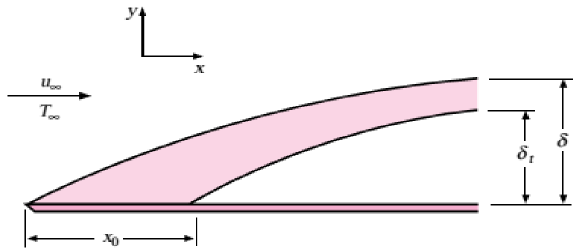
Conditions (a) to (d) may be fitted to a cubic polynomial as in the case of the velocity profile, so that.

$$\boldsymbol{\frac{\theta}{\theta\infty} = \frac{T-Tw}{T\infty-Tw} = \frac{3}{2}\frac{y}{\delta t} - \frac{1}{2}\left(\frac{y}{\delta t}\right)^3}$$

Where θ = T – Tw

δt, the thermal-boundary-layer thickness

Flat plate with unheated starting length



$$\frac{\delta}{\delta t} = \zeta = \frac{1}{1.026}Pr^{-\frac{1}{3}}\left[1-\left(\frac{x0}{x}\right)^{\frac{3}{4}}\right]^{\frac{1}{3}} \qquad (***)$$

Hydrodynamic and thermal boundary layers on a flat plate. Heating starts at *x* = *x*0.

When the plate is heated over the entire length, x0 = 0, and

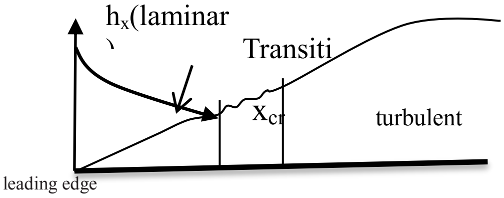
$$\boldsymbol{\frac{\delta}{\delta t} = \zeta = \frac{1}{1.026}Pr^{-1/3}}$$

$\zeta < 1$ this assumption is satisfactory for fluids having Prandtl numbers greater than about 0.7. Fortunately, most gases and liquids fall within this category. Liquid metals are a notable exception, however ,since they have Prandtl numbers of the order of 0.01.

$$h = \frac{-k\frac{\partial T}{\partial y}]_{wall}}{Tw - T\infty} = \frac{3}{2}\frac{k}{\delta t} = \frac{3}{2}\frac{k}{\zeta\delta}$$

Substituting for the hydrodynamic-boundary-layer thickness from Equatio and using Equation (***)

$$h_x = 0.332\ k\ Pr^{\frac{1}{3}}(\frac{u\infty}{\vartheta x})^{1/2}[1 - \left(\frac{x0}{x}\right)^{\frac{3}{4}}]^{-1/3}$$



(h_x)local values of the heat-transfer coefficient in terms of the distance from the leading edge of the plate and the fluid properties.

The equation may be nondimensionalized by multiplying both sides by x/k, producing the dimensionless group on the left side .

$$h_x\frac{x}{k} = 0.332\ k * \frac{x}{k}\ Pr^{\frac{1}{3}}(\frac{u\infty}{\vartheta x})^{1/2}[1 - \left(\frac{x0}{x}\right)^{\frac{3}{4}}]^{-1/3}$$

$Nu_x = \frac{h_x\, x}{k}$ Nu_x Called the Nusselt number after Wilhelm Nusselt, who made significant contributions to the theory of convection heat transfer. Finally.

Flat plate heated over its entire length, (x0 = 0) constant temperature

$$Nu_x = 0.332\ Pr^{\frac{1}{3}}Re_x^{\frac{1}{2}} \quad laminar\ \text{(local Nusselt No.)}\quad Pr \le 0.6$$

The corresponding relations for turbulent flow

$$Nu_x = 0.0296\ Pr^{\frac{1}{3}}Re_x^{0.8} \quad \text{turbulent (local Nusselt No.)} \quad 0.6 \le Pr \le 60$$

Flat plate with unheated starting length(x=x0) constant temperature

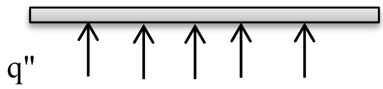
$$\boldsymbol{Nu_x = \frac{0.332\ Pr^{\frac{1}{3}} Re_x{}^{1/2}}{[1-(\frac{x0}{x})^{\frac{3}{4}}]^{1/3}}}$$ laminar

$$Nu_x = \frac{0.0296\ Pr^{\frac{1}{3}} Re_x{}^{0.8}}{[1-(\frac{x0}{x})^{9/10}]^{1/9}}$$ turbulent

Constant Heat Flux:

The constant-heat-flux case it can be shown that the local Nusselt number is given by

$$Nu_x = \frac{hx}{k} = 0.453 Re_x^{\frac{1}{2}}\ Pr^{\frac{1}{3}} \qquad laminar$$



which may be expressed in terms of the wall heat flux and temperature difference as

$$Nu_x = \frac{q_W x}{k(Tw - T\infty)}$$

$$Nu_x = \frac{hx}{k} = 0.0308 Re_x^{0.8}\ Pr^{\frac{1}{3}} \qquad turbulent$$

The Average Heat-Transfer Coefficient and Nusselt Number

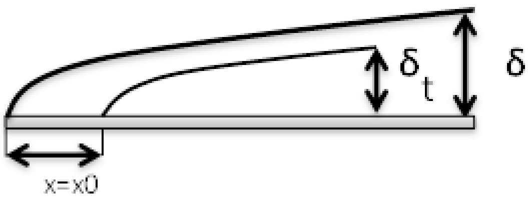
For a plate where heating starts at x0 =0

The average heat-transfer coefficient and Nusselt number may be obtained by integrating over the length of the plate: $\bar{h} = \frac{\int_0^L h_x\ dx}{\int_0^L dx} = 2h_{x=L}$

For a plate where heating starts at x = x0

It can be shown that the average heat transfer coefficient can be expressed as :

$$\boldsymbol{\frac{\bar{h}_{x0-L}}{h_{x=L}} = 2L \frac{1-(\frac{x0}{L})^{3/4}}{L-x0}}$$



In this case, the total heat transfer for the plate would be

$$\boldsymbol{q_{total} = \bar{h}_{x0-L}(L - x0)(Tw - T\infty)}$$