## 3.6 Laminar flow in a concentric annulus

The flow of non-Newtonian fluids through concentric and eccentric annuli represents an idealisation of several industrially important processes. One important example is in oil well drilling where a heavy drilling mud is circulated through the annular space around the drill pipe in order to carry the drilling debris to the surface. These drilling muds are typically either Bingham plastic or power-law type fluids. Other examples include the extrusion of plastic tubes and pipes in which the molten polymer is forced through an annular die, and the flow in double-pipe heat exchangers. In all these applications, it is often required to predict the frictional pressure gradient to sustain a fixed flow rate or vice versa. In this section, the isothermal, steady and fully-developed flow of power-law and Bingham plastic fluids in concentric annulus is analysed and appropriate expressions and/or charts are presented which permit the calculation of pressure gradient for a given application.

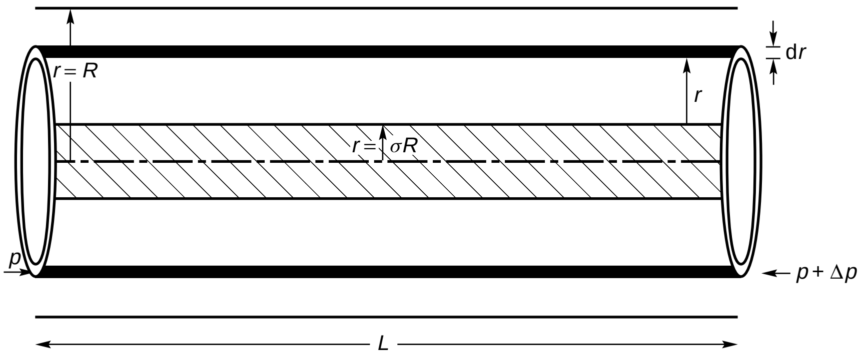


**Figure 3.17** *Flow in a concentric annulus*

The calculation of the velocity distribution and the mean velocity of a fluid flowing through an annulus of outer radius $R$ and inner radius $\sigma R$ is more complex than that for flow in a pipe or between two parallel planes (Figure 3.17), though the force balance on an element of fluid can be written in a manner similar to that used in previous sections. If the pressure changes by an amount $\Delta p$ as a consequence of friction in a length $L$ of annulus, the resulting force can be equated to the shearing force acting on the fluid. Consider the flow of the fluid situated at a distance not greater than $r$ from the centreline of the pipe. The shear force acting on this fluid comprises two parts: one is the drag on its outer surface ($r = R$) which can be expressed in terms of the shear stress in the fluid at that location; the other contribution is the drag occurring at the inner (solid) boundary of the annulus, i.e. at $r = \sigma R$. This component cannot be estimated at present, however. Alternatively, this difficulty can be obviated by considering the equilibrium of a thin ring of fluid of radius $r$ and thickness d$r$ (Figure 3.17). The pressure force acting on this fluid element is:

$$2\pi r \mathrm{d}r\{p - (p + \Delta p)\}$$

The only other force acting on the fluid element in the $z$-direction is that arising from the shearing on both surfaces of the element. Note that, not only will the shear stress change from $r$ to $r + \mathrm{d}r$ but the surface area over which shearing occurs will also depend upon the value of $r$. The net force can be written as:

$$2\pi r\,L\cdot\tau_{rz}\;{}_{r+\mathrm{d}r} - 2\pi r\,L\,\tau_{rz}\;{}_{r}$$

At equilibrium therefore:

$$2\pi r\mathrm{d}r\left(\frac{-\Delta p}{L}\right.\quad L = 2\pi L\quad r\,\tau_{rz}\;{}_{r+\mathrm{d}r} - r\tau_{rz}\;{}_{r}$$

or $$\frac{r\tau_{rz}\;{}_{r+\mathrm{d}r} - r\tau_{rz}\;{}_{r}}{\mathrm{d}r} = r\left(\frac{-\Delta p}{L}\right.$$

Now taking limits as $\mathrm{d}r \to 0$, it becomes

$$\frac{\mathrm{d}}{\mathrm{d}r}(r\tau_{rz}) = r\left(\frac{-\Delta p}{L}\right. \tag{3.75}$$

The shear stress distribution across the gap is obtained by integration:

$$\tau_{rz} = \frac{r}{2}\left(\frac{-\Delta p}{L}\right.\quad + \frac{C_1}{r} \tag{3.76}$$

Because of the no-slip boundary condition at both solid walls, i.e. at $r = \sigma R$ and $r = R$, the velocity must be maximum at some intermediate point, say at $r = \lambda R$. Then, for a fluid without a yield stress, the shear stress must be zero at this position and for a viscoplastic fluid, there will be a plug moving *en masse*. Equation (3.76) can therefore be re-written:

$$\tau_{rz} = \left(\frac{-\Delta p}{L}\right.\quad \frac{R}{2}\left(\xi - \frac{\lambda^2}{\xi}\right. \tag{3.77}$$

where $\xi = r/R$, the dimensionless radial coordinate.

### 3.6.1 Power-law fluids

For this flow, the power-law fluid can be written as:

$$\tau_{rz} = -m\quad \frac{\mathrm{d}V_z}{\mathrm{d}r}^{\;n-1}\left(\frac{\mathrm{d}V_z}{\mathrm{d}r}\right. \tag{3.78}$$

It is important to write the equation in this form whenever the sign of the velocity gradient changes within the flow field. In this case, $(\mathrm{d}V_z/\mathrm{d}r)$ is positive for $\sigma \le \xi \le \lambda$ and negative for $\lambda \le \xi \le 1$. Now equation (3.78) can be substituted in equation (3.77) and integrated to obtain

$$V_{z_i} = R\left(\frac{-\Delta p}{L}\cdot\frac{R}{2m}\right)^{1/n}\int_{\sigma}^{\xi}\left(\frac{\lambda^2}{x} - x\right)^{1/n}\mathrm{d}x \tag{3.79a}$$

$$\sigma \le \xi \le \lambda$$

$$V_{z_o} = R\left(\frac{-\Delta p}{L}\cdot\frac{R}{2m}\right)^{1/n}\int_{\xi}^{1}\left(x - \frac{\lambda^2}{x}\right)^{1/n}\mathrm{d}x \tag{3.79b}$$

$$\lambda \le \xi \le 1$$

where subscripts '$i$' and '$o$' denote the inner ($\sigma \le \xi \le \lambda$) and outer ($\lambda \le \xi \le 1$) regions respectively and $x$ is a dummy variable of integration. The no-slip boundary conditions at $\xi = \sigma$ and $\xi = 1$ have been incorporated in equation (3.79). Clearly, the value of $\lambda$ is evaluated by setting $V_{z_i} = V_{z_0}$ at $\xi = \lambda$, i.e.

$$\int_{\sigma}^{\lambda}\left(\frac{\lambda^2}{x} - x\right)^{1/n}\mathrm{d}x = \int_{\lambda}^{1}\left(x - \frac{\lambda^2}{x}\right)^{1/n}\mathrm{d}x \tag{3.80}$$

The volumetric flow rate of the fluid, $Q$, is obtained as:

$$\begin{aligned} Q &= 2\pi\int_{\sigma R}^{R} rV_z\,\mathrm{d}r = 2\pi R^2\int_{\sigma}^{1}\xi V_z\,\mathrm{d}\xi \\ &= 2\pi R^3\left(\frac{-\Delta p}{L}\cdot\frac{R}{2m}\right)^{1/n}\left[\int_{\sigma}^{\lambda}\xi V_{z_i}\mathrm{d}\xi + \int_{\lambda}^{1}\xi V_{z_o}\,\mathrm{d}\xi\right] \end{aligned} \tag{3.81}$$

Clearly, equations (3.80) and (3.81) must be solved and integrated simultaneously to eliminate $\lambda$ and to evaluate the volumetric rate of flow of liquid, $Q$. Analytical solutions are possible only for integral values of $(1/n)$, i.e. for $n = 1, 0.5, 0.33, 0.25$, etc. Thus, Fredrickson and Bird [1958] evaluated the integral in equation (3.81) for such values of $n$ and, by interpolating the results for the intermediate values of power law index, they presented a chart relating non-dimensional flowrate, pressure drop, $\sigma$ and $n$. However, the accuracy of their results deteriorates rapidly with decreasing values of $n$ and/or $(1-\sigma) \ll 1$, i.e. with narrowing annular region. Subsequently, however, Hanks and Larsen [1979] were able to evaluate the volumetric flow rate, $Q$, analytically and their final expression is:

$$Q = \frac{n\pi R^3}{(3n+1)}\left(\frac{-\Delta p}{L}\cdot\frac{R}{2m}\right)^{1/n}\{(1-\lambda^2)^{(n+1)/n} - \sigma^{(n-1)/n}(\lambda^2-\sigma^2)^{(n+1)/n}\} \tag{3.82}$$

**Table 3.2** *Values of λ computed from equation (3.80) [Hanks and Larson, 1979]*

| | $\sigma$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $n$ | 0.10 | 0.20 | 0.30 | 0.40 | 0.50 | 0.60 | 0.70 | 0.80 | 0.90 |
| 0.10 | 0.3442 | 0.4687 | 0.5632 | 0.6431 | 0.7140 | 0.7788 | 0.8389 | 0.8954 | 0.9489 |
| 0.20 | 0.3682 | 0.4856 | 0.5749 | 0.6509 | 0.7191 | 0.7818 | 0.8404 | 0.8960 | 0.9491 |
| 0.30 | 0.3884 | 0.4991 | 0.5840 | 0.6570 | 0.7229 | 0.7840 | 0.8416 | 0.8965 | 0.9492 |
| 0.40 | 0.4052 | 0.5100 | 0.5912 | 0.6617 | 0.7259 | 0.7858 | 0.8426 | 0.8969 | 0.9493 |
| 0.50 | 0.4193 | 0.5189 | 0.5970 | 0.6655 | 0.7283 | 0.7872 | 0.8433 | 0.8972 | 0.9493 |
| 0.60 | 0.4312 | 0.5262 | 0.6018 | 0.6686 | 0.7303 | 0.7884 | 0.8439 | 0.8975 | 0.9494 |
| 0.70 | 0.4412 | 0.5324 | 0.6059 | 0.6713 | 0.7319 | 0.7893 | 0.8444 | 0.8977 | 0.9495 |
| 0.80 | 0.4498 | 0.5377 | 0.6093 | 0.6735 | 0.7333 | 0.7902 | 0.8449 | 0.8979 | 0.9495 |
| 0.90 | 0.4572 | 0.5422 | 0.6122 | 0.6754 | 0.7345 | 0.7909 | 0.8452 | 0.8980 | 0.9495 |
| 1.00 | 0.4637 | 0.5461 | 0.6147 | 0.6770 | 0.7355 | 0.7915 | 0.8455 | 0.8981 | 0.9496 |

The only unknown now remaining is λ, which locates the position where the velocity is maximum. Table 3.2 presents the values of λ for a range of values of $\sigma$ and $n$.

### Example 3.10

A polymer solution exhibits power-law behaviour with $n = 0.5$ and $m = 3.2\,\text{Pa·s}^{0.5}$. Estimate the pressure gradient required to maintain a steady flow of $0.3\,\text{m}^3/\text{min}$ of this polymer solution through the annulus between a 10 mm and a 20 mm diameter tube.

### Solution

$$\text{Here,}\quad R = \frac{20}{2} \times 10^{-3} = 0.01\,\text{m}$$

$$\sigma R = \frac{10}{2} \times 10^{-3} = 0.005\,\text{m}$$

$$\text{or}\quad \sigma = 0.5$$

From Table 3.2, for $\sigma = 0.5$ and $n = 0.5$, $\lambda = 0.728$
Substituting these values in equation (3.82)

$$\left(\frac{0.3}{60}\right) = \frac{(0.5)(3.14)(0.01)^3}{(3 \times 0.5 + 1)}\left(\frac{-\Delta p}{L}\right)^2\left(\frac{0.01}{2 \times 3.2}\right)^2$$
$$\times \{(1 - 0.728^2)^{(0.5+1)/0.5} - 0.5^{(0.5-1)/0.5}(0.728^2 - 0.5^2)^{(0.5+1)/0.5}\}$$

and solving: $\dfrac{-\Delta p}{L} = 169\,\text{kPa/m}$