Supporting Australian Mathematics Project

A guide for teachers – Years 11 and 12

Calculus: Module 15

The calculus of trigonometric functions

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The calculus of trigonometric functions – A guide for teachers (Years 11–12)

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The calculus of trigonometric functions

Assumed knowledge

The content of the modules:

- *Trigonometric functions and circular measure*
- *Introduction to differential calculus*
- *Integration*.

Motivation

It is an interesting exercise to sit back and think about how we have developed the topic of trigonometry during the earlier secondary school years.

We commenced by looking at ratios of sides in a right-angled triangle. This enabled us to find unknown sides and angles. We extended this to include non-right-angled triangles using the sine and cosine rules.

In the module *Trigonometric functions and circular measure*, we redefined the sine and cosine functions in terms of the coordinates of points on the unit circle. This enabled us to define the sine and cosine of angles greater than 90° and to plot the graphs of the trigonometric functions and discover their periodic nature.

The sine and cosine functions are used to model periodic phenomena in nature, such as waves, tides and signals. Indeed, these functions are used to model all sorts of oscillatory motion arising in a range of subjects, including economics and ecology.

In this module, we continue this development by applying the ideas and techniques of calculus to the trigonometric functions. For example, if we wish to analyse the motion of a particle modelled by a trigonometric function, we can use calculus to find its velocity and acceleration.

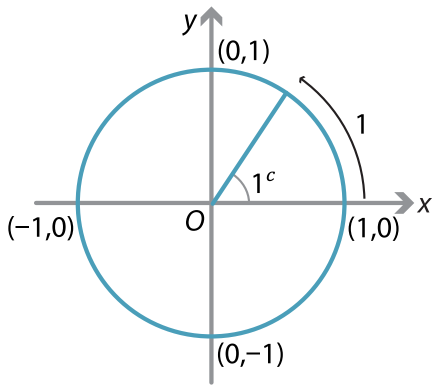
The simplicity of the results obtained by doing this is amazing and has wide-ranging impact in physics and electrical engineering, and indeed in any area in which periodic motion is being modelled.

Content

Review of radian measure

We saw in the module *Trigonometric functions and circular measure* that angles can be naturally defined using arc length.

We define 1 radian (written as 1^c) to be the angle subtended in the unit circle by an arc length of one unit.



Since the circumference of the unit circle is 2π and the angle in one revolution is 360°, we can relate the two units by $2\pi^c = 360°$ or

$$\pi = 180°.$$

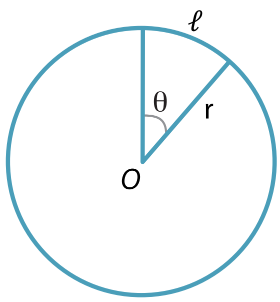
(As usual, we will drop the superscript c when it is clear that the angle under discussion is in radians.) Many commonly occurring angles can be expressed in radians as fractions of π. For example, $60° = \frac{\pi}{3}$ and $330° = \frac{11\pi}{6}$.

We have also seen that the arc length ℓ of a sector of a circle of radius r, containing an angle θ (in radians), is given by

$$\ell = r\theta,$$

while the area A of the sector is given by

$$A = \frac{1}{2}r^2\theta.$$



An important limit

In order to apply calculus to the trigonometric functions, we will need to evaluate the fundamental limit

$$\lim_{x \to 0} \frac{\sin x}{x},$$

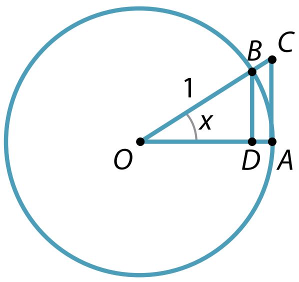
which arises when we apply the definition of the derivative of $f(x) = \sin x$.

It must be stressed that, from here on in this module, we are measuring x in radians.

You can see that attempting to substitute $x = 0$ into $\frac{\sin x}{x}$ is fruitless, since we obtain the indeterminate form 'zero over zero'. On the other hand, if we try substituting say $x = 0.1$ (in radians), then the calculator gives $0.998\dots$, and if we try $x = 0.01$, we get $0.99998\dots$.

We guess that, as x approaches 0, the value of $\frac{\sin x}{x}$ approaches 1. We will now give a nice geometric proof of this fact.

Consider a sector OAB of the unit circle containing an acute angle x, as shown in the following diagram. Drop the perpendicular BD to OA and raise a perpendicular at A to meet OB produced at C. Since $OA = OB = 1$, we have $BD = \sin x$ and $AC = \tan x$.



The area of the sector OAB is $\frac{1}{2}x$. The sector OAB clearly contains the triangle OAB and is contained inside the triangle OAC. Hence, comparing their areas, we have

$$\begin{aligned}
\text{Area } \triangle OAB \le \frac{1}{2}x \le \text{Area } \triangle OAC \quad &\Longrightarrow \quad \frac{1}{2}OA \cdot BD \le \frac{1}{2}x \le \frac{1}{2}OA \cdot AC \\
&\Longrightarrow \quad \frac{1}{2}\sin x \le \frac{1}{2}x \le \frac{1}{2}\tan x \\
&\Longrightarrow \quad \sin x \le x \le \tan x.
\end{aligned}$$

Now, since $\tan x = \frac{\sin x}{\cos x}$ and since $\sin x > 0$ for $0 < x < \frac{\pi}{2}$, we can divide both inequalities by $\sin x$ to obtain

$$1 \le \frac{x}{\sin x} \le \frac{1}{\cos x}.$$

Since $\cos x$ approaches 1 as x approaches 0, we see that

$$\frac{x}{\sin x} \to 1 \quad \text{as} \quad x \to 0.$$

Taking the reciprocal, it follows that

$$\frac{\sin x}{x} \to 1 \quad \text{as} \quad x \to 0.$$

(Here we have used the pinching theorem and the algebra of limits, as discussed in the module *Limits and continuity*.)

Thus we can write

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Note that this means that, if the angle x is small, then $\sin x \approx x$. This fact is often used by physicists when analysing such things as a simple pendulum with small angle.

Exercise 1

Using the limit above, prove that

$$\lim_{x\to 0} \frac{\tan x}{x} = 1.$$

Other related limits can be found by manipulating this basic limit.

Example

Find

$$\lim_{x\to 0} \frac{\sin 2x}{3x}.$$

Solution

We can write

$$\frac{\sin 2x}{3x} = \frac{\sin 2x}{2x} \times \frac{2}{3}.$$

If we let $u = 2x$, then as $x \to 0$, we have $u \to 0$. Hence

$$\lim_{x\to 0} \frac{\sin 2x}{3x} = \frac{2}{3} \lim_{u\to 0} \frac{\sin u}{u} = \frac{2}{3}.$$

Exercise 2

Find

$$\lim_{x\to 0} \frac{\sin 3x + \sin 7x}{5x}.$$

Exercise 3

a Show that $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$.

b Deduce that $\cos x \approx 1 - \frac{1}{2}x^2$, for small x.

Differentiating trigonometric functions

The derivative of sine

Since the graph of $y = \sin x$ is a smooth curve, we would like to find the gradient of the tangent to the curve at any point on it.

Before doing this, we derive a useful trigonometric identity that will assist us.

Using the compound-angle formulas, we have

$$\begin{aligned}\sin(A+B) - \sin(A-B) &= \sin A \cos B + \cos A \sin B - (\sin A \cos B - \cos A \sin B)\\ &= 2\cos A \sin B.\end{aligned}$$

If we put $C = A+B$ and $D = A-B$, we can add these equations to obtain $A = \frac{1}{2}(C+D)$ and subtract them to obtain $B = \frac{1}{2}(C-D)$. Substituting these back, we obtain the **sine difference formula**:

$$\sin C - \sin D = 2\cos\Big(\frac{C+D}{2}\Big)\sin\Big(\frac{C-D}{2}\Big).$$

To find the derivative of $\sin x$, we return to the first principles definition of the derivative of $y = f(x)$:

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Substituting $y = \sin x$, we have

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}.$$

Applying the sine difference formula, we have

$$\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}=\lim_{h\to 0}\frac{2\cos\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}.$$

We can put $u=\frac{h}{2}$. Then $u\to 0$ as $h\to 0$. So the limit becomes

$$\begin{aligned}\frac{dy}{dx}&=\lim_{u\to 0}\frac{2\cos(x+u)\sin u}{2u}\\&=\left(\lim_{u\to 0}\cos(x+u)\right)\times\left(\lim_{u\to 0}\frac{\sin u}{u}\right)\\&=\cos x\times 1\\&=\cos x.\end{aligned}$$

We can thus conclude that

$$\frac{d}{dx}(\sin x)=\cos x.$$

This result is both simple and surprising, and students need to commit it to memory.

The derivation above involved a number of ingredients and is often difficult for students the first time through.

Derivatives of other trigonometric functions

Now that the derivative of sine is established, we can use the standard rules of calculus — the chain, product and quotient rules — to proceed.

Since $\cos x=\sin\left(x+\frac{\pi}{2}\right)$, we can apply the chain rule to see that

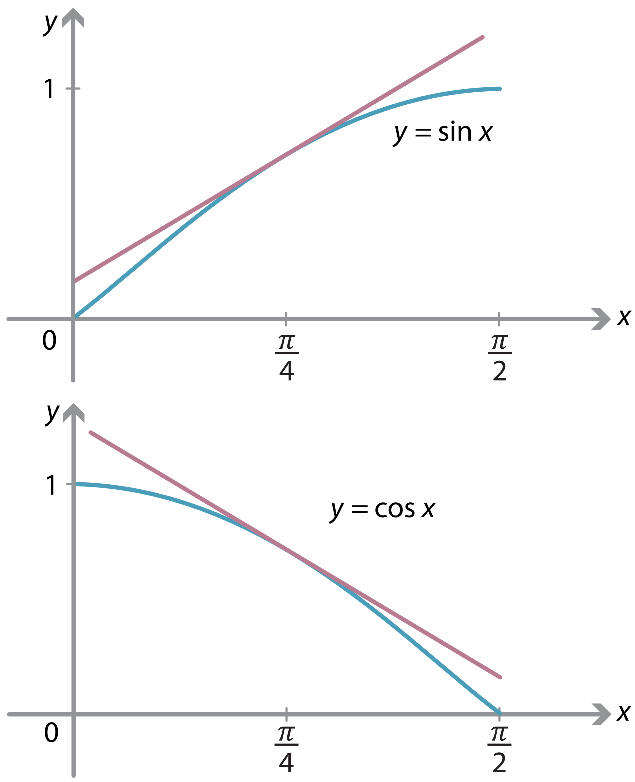
$$\begin{aligned}\frac{d}{dx}(\cos x)&=\frac{d}{dx}\left(\sin\left(x+\frac{\pi}{2}\right)\right)\\&=\cos\left(x+\frac{\pi}{2}\right)\\&=-\sin x.\end{aligned}$$

Thus

$$\frac{d}{dx}(\cos x)=-\sin x.$$

This is also a simple and surprising result that needs to be committed to memory.

The following graphs illustrate what is happening geometrically. If we draw the tangent to the curve $y = \sin x$ at a point with $0 < x < \frac{\pi}{2}$, then the tangent clearly has positive gradient, while a tangent to $y = \cos x$, in the same range, clearly has negative gradient.



The derivative of $\cos x$ can also be found by using first principles.

Exercise 4

a Show that

$$\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right).$$

b Show by first principles that

$$\frac{d}{dx}(\cos x) = -\sin x.$$

Exercise 5

By writing $\tan x = \dfrac{\sin x}{\cos x}$ and applying the quotient rule, prove that

$$\frac{d}{dx}(\tan x) = \sec^2 x.$$

Students need to remember the derivatives of sin, cos and tan.

The rules of calculus can also be used to find the derivatives of the reciprocal functions.

Exercise 6

Show that

a $\dfrac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

b $\dfrac{d}{dx}(\sec x) = \sec x \tan x$

c $\dfrac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x.$

These three derivatives need not be committed to memory.

Further examples

Example

Use the rules of calculus to differentiate each of the following functions with respect to x:

1. $4\sin(2x^2)$
2. $x\cos(2x)$
3. $e^{3x}\tan(4x)$.

Solution

1. $\dfrac{d}{dx}\big(4\sin(2x^2)\big) = 16x\cos(2x^2)$
2. $\dfrac{d}{dx}\big(x\cos(2x)\big) = -2x\sin(2x) + \cos(2x)$
3. $\dfrac{d}{dx}\big(e^{3x}\tan(4x)\big) = 4e^{3x}\sec^2(4x) + 3e^{3x}\tan(4x)$.

Exercise 7

Show that

$$\frac{d}{dx}\log_e\Big(\frac{1+\sin x}{\cos x}\Big) = \sec x.$$

Applications of the derivatives

Armed with the ability to differentiate trigonometric functions, we can now find the equations of tangents to trigonometric functions and find local maxima and minima.

Example

1 Find the equation of the tangent to the curve $y = 2\sin x + \cos 2x$ at the point $x = \pi$.

2 Find the minimum value of $y = 2\sin x + \cos 2x$ in the interval $0 \leq x \leq 2\pi$.

Solution

1 The gradient of the tangent is given by $\dfrac{dy}{dx} = 2\cos x - 2\sin 2x = -2$ at $x = \pi$. The y-value at this point is 1. Hence, the equation of the tangent is

$$y - 1 = -2(x - \pi) \quad \text{or, equivalently,} \quad y + 2x = 1 + 2\pi.$$

2 Since the function is continuous, the minimum will occur either at an end point of the interval $0 \leq x \leq 2\pi$ or at a stationary point. The y-value at each endpoint is 1. To find the stationary points, we solve $\dfrac{dy}{dx} = 0$. This gives $\cos x = \sin 2x$. To proceed, we use a double-angle formula:

$$\begin{aligned} \cos x = \sin 2x \quad &\implies \quad \cos x = 2\sin x \cos x \\ &\implies \quad (1 - 2\sin x)\cos x = 0. \end{aligned}$$

Hence $\cos x = 0$ or $\sin x = \frac{1}{2}$. The solutions in the range $0 \leq x \leq 2\pi$ are

$$x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}.$$

The smallest y-value, which is -3, occurs at $x = \dfrac{3\pi}{2}$. Hence the minimum value is -3.

Exercise 8

Show that the function $y = \dfrac{\sin x}{3 + 4\cos x}$ is increasing in each interval in which the denominator is not zero.

Exercise 9

Suppose an isosceles triangle has two equal sides of length a and equal base angles θ. Show that the perimeter of the triangle is $2a(1 + \cos\theta)$. Deduce that, of all isosceles triangles with fixed perimeter, the triangle of largest area is equilateral.

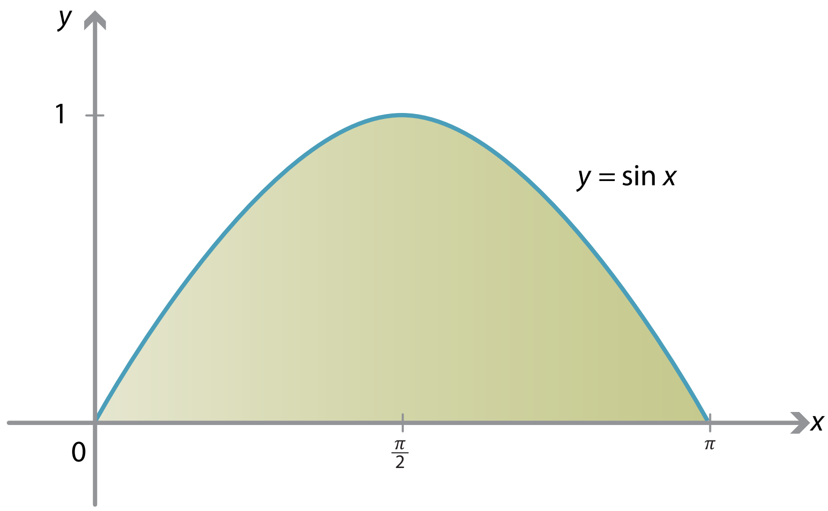
Integrating trigonometric functions

Since integration is the reverse of differentiation, we have immediately that

$$\int \cos x \, dx = \sin x + C \qquad \text{and} \qquad \int \sin x \, dx = -\cos x + C.$$

Thus, for example, we can find the area under the sine curve between $x = 0$ and $x = \pi$, as shown on the following graph.

$$\text{Area} = \int_0^{\pi} \sin x \, dx = \big[-\cos x\big]_0^{\pi} = 2.$$



More generally, since

$$\frac{d}{dx}\sin(ax+b) = a\cos(ax+b) \quad \text{and} \quad \frac{d}{dx}\cos(ax+b) = -a\sin(ax+b),$$

we obtain, for $a \neq 0$,

$$\int \cos(ax+b)\, dx = \frac{1}{a}\sin(ax+b) + C \quad \text{and} \quad \int \sin(ax+b)\, dx = -\frac{1}{a}\cos(ax+b) + C.$$

From $\dfrac{d}{dx}(\tan x) = \sec^2 x$, we obtain

$$\int \sec^2 x \, dx = \tan x + C.$$

Example

Find

1 $\displaystyle\int_0^{\frac{\pi}{2}} (1 + \cos 2x)\, dx$

2 $\displaystyle\int_0^{\frac{\pi}{6}} (\sin 3x + \sec^2 2x)\, dx.$

Solution

1 $\displaystyle\int_0^{\frac{\pi}{2}} (1 + \cos 2x)\, dx = \Big[x + \frac{1}{2}\sin 2x\Big]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$

2 $\displaystyle\int_0^{\frac{\pi}{6}} (\sin 3x + \sec^2 2x)\, dx = \Big[-\frac{1}{3}\cos 3x + \frac{1}{2}\tan 2x\Big]_0^{\frac{\pi}{6}} = \frac{1}{3} + \frac{\sqrt{3}}{2}.$

Exercise 10

a Find $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (\sin 2x + \cos 3x)\, dx.$

b Differentiate $x \sin x$, and hence find $\displaystyle\int_0^{\frac{\pi}{2}} x \cos x\, dx.$

c Use the identity $1 + \tan^2 x = \sec^2 x$ to find $\displaystyle\int \tan^2 x\, dx.$

Special integrals

The two integrals

$$\int \cos^2\theta\, d\theta \qquad \text{and} \qquad \int \sin^2\theta\, d\theta$$

require some special attention. They are handled in similar ways.

To proceed, we make use of two trigonometric identities (a double-angle formula and the Pythagorean identity):

$$\cos^2\theta - \sin^2\theta = \cos 2\theta \qquad \text{and} \qquad \cos^2\theta + \sin^2\theta = 1.$$

Adding these two identities, we have $2\cos^2\theta = 1 + \cos 2\theta$, and so we can replace $\cos^2\theta$ in the integral with $\frac{1}{2}(1 + \cos 2\theta)$. Thus

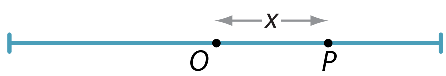
$$\int \cos^2\theta\, d\theta = \int \frac{1}{2}(1 + \cos 2\theta)\, d\theta = \frac{1}{2}\Big(\theta + \frac{1}{2}\sin 2\theta\Big) + C.$$

Exercise 11

Find $\int \sin^2\theta\, d\theta$. (Follow the method used for $\cos^2\theta$, but subtract the two identities rather than adding them.)

Simple harmonic motion

Simple harmonic motion (SHM) is a special case of motion in a straight line which occurs in several examples in nature. A particle P is said to be undergoing simple harmonic motion when it moves backwards and forwards about a fixed point (the centre of motion) so that its acceleration is directed back towards the centre of motion and proportional to its displacement from the centre.



Hence the displacement x of the particle P will satisfy the equation

$$\frac{d^2x}{dt^2} = -n^2 x,$$

where n is a positive constant.

This is an example of a second order differential equation. It can be shown that the general solution to this equation is $x(t) = A\sin nt + B\cos nt$, where A and B are constants.

In the case where the particle starts at the origin, so $x = 0$ when $t = 0$, we have $B = 0$ and so the function $x(t) = A\sin nt$ is a solution to the differential equation. We can easily check this:

$$x(t) = A\sin nt \quad\Longrightarrow\quad \frac{d^2x}{dt^2} = -n^2 A\sin nt = -n^2 x.$$

In the general case, since any trigonometric expression of the form $A\sin\theta + B\cos\theta$ can written in the form $C\sin(\theta+\alpha)$, we can write the general solution as

$$x(t) = C\sin(nt+\alpha),$$

where C and α are constants. The constant α is called the **phase shift** of the motion (and as we saw above can be taken as 0 if the particle begins at the origin). From our knowledge of the trigonometric functions, we see that the amplitude of the motion is C and the period is $\frac{2\pi}{n}$.

We next derive a formula for the velocity of the particle, with the help of a very useful expression for acceleration: $\dfrac{d^2x}{dt^2} = \dfrac{1}{2}\dfrac{d}{dx}(v^2)$. We write the differential equation as

$$\frac{1}{2}\frac{d}{dx}(v^2) = -n^2 x$$

and integrate with respect to x to obtain

$$v^2 = K - n^2 x^2,$$

where K is a constant. If the amplitude of the motion is C, then when $x = C$ the velocity is 0, and so $K = n^2C^2$. Hence, we have

$$v^2 = n^2(C^2 - x^2).$$

Example

A particle is moving in simple harmonic motion. Find a formula for the displacement $x(t)$ of the particle (with x in metres and t in seconds) given that:

- the period of the motion is 16 seconds
- the particle passes through the centre of oscillation when $t = 2$
- the particle has a velocity of 2π m/s when $t = 4$.

What is the amplitude of the motion?

Solution

The general equation for simple harmonic motion is $x = C\sin(nt + \alpha)$. Since the period is 16, we have $\dfrac{2\pi}{n} = 16$, giving $n = \dfrac{\pi}{8}$.

Also, since $x = 0$ when $t = 2$, we have $C\sin(2n + \alpha) = 0$. As $C \neq 0$, we may take $2n + \alpha = 0$, and so $\alpha = -2n = -\dfrac{\pi}{4}$.

Finally, the velocity is given by

$$v = \frac{dx}{dt} = Cn\cos(nt + \alpha) = \frac{C\pi}{8}\cos\Big(\frac{\pi}{8}t - \frac{\pi}{4}\Big).$$

Now, $v = 2\pi$ when $t = 4$, giving $\dfrac{C\pi}{8} \cdot \dfrac{1}{\sqrt{2}} = 2\pi$, and so $C = 16\sqrt{2}$.

Thus the equation for displacement is

$$x = 16\sqrt{2}\sin\Big(\frac{\pi}{8}t - \frac{\pi}{4}\Big)$$

and the amplitude is $16\sqrt{2}$.

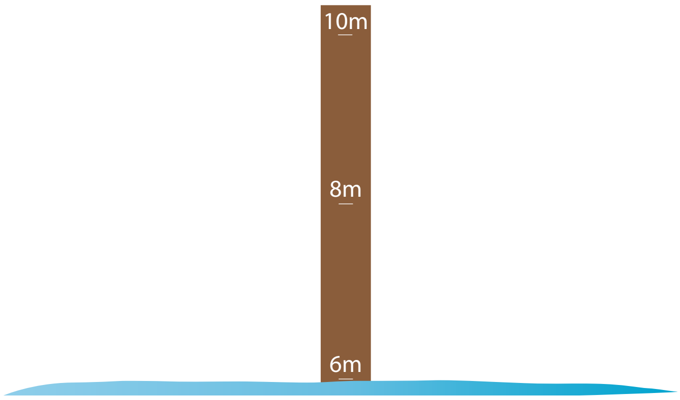
Exercise 12

Find the speed of a particle moving in SHM when it is passing through the centre, if its period is $\pi\sqrt{2}$ seconds and its amplitude is 2 metres.

Example

In a certain bay, there is a low tide of 6 metres at 1am and a high tide of 10 metres at 8am. Assuming that the tide motion is simple harmonic, find an expression for the height at time t after 1am, and find the first time after 1am when the tide is 9 metres.

Solution



Let t be the time in hours since 1am, and let x be the height of the tide in metres at time t. Since the centre of the motion is at 8 m and the amplitude is 2 m, we can express the height as $x = 8 + 2\sin(nt + \alpha)$.

The period is 14 hours, so $\dfrac{2\pi}{n} = 14$, giving $n = \dfrac{\pi}{7}$. Also, since $x = 6$ when $t = 0$, we may take $\alpha = -\dfrac{\pi}{2}$. Hence the height of the tide is

$$x = 8 + 2\sin\Big(\frac{\pi}{7}t - \frac{\pi}{2}\Big).$$

(We can check that when $t = 7$, $x = 10$, as expected.)

Now, when $x = 9$, we have $\sin\Big(\dfrac{\pi}{7}t - \dfrac{\pi}{2}\Big) = \dfrac{1}{2}$. The smallest solution is $t = \dfrac{14}{3}$, that is, 4 hours 40 minutes after 1am.

So the required time is 5:40am.

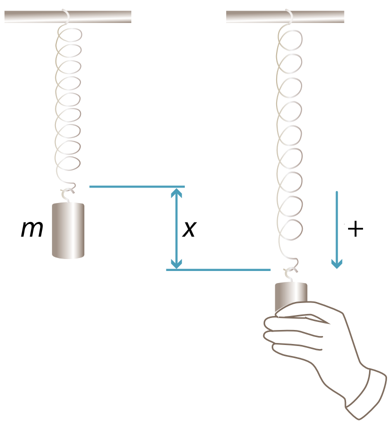
Hooke's law

An inextensible string is one which can bear a mass without altering its length. In practice, all strings are extensible, however the extension is usually very negligible. In the case of a string which is extensible, Hooke's law provides a simple relationship between the tension in the string and the extension it experiences. It states that the tension in an elastic string (or spring) is directly proportional to the extension of the string beyond its natural length.

Thus, if T is the tension in the spring, then $T = kx$, where x is the extension and k is a positive constant, sometimes called the **stiffness constant** for the spring.

Suppose that we have a particle of mass m attached to the bottom of a vertical spring with stiffness constant k. Since the system is stationary, the tension in the spring is given by mg, where g is the acceleration due to gravity. By Hooke's law, the tension in the spring is also equal to ke, where e is the extension of the natural length of the spring after the mass is added. We let the position of the particle at this stage be O, and then further depress the mass by a displacement x from O and release it. The tension T in the spring is now $T = k(e + x) = mg + kx$. The resultant downward force is then

$$F = mg - T = -kx.$$



This force produces an acceleration: by Newton's second law of motion, $F = m\dfrac{d^2x}{dt^2}$. So we have

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x.$$

This is the differential equation for simple harmonic motion with $n^2 = \dfrac{k}{m}$. Hence, the period of the motion is given by $\dfrac{2\pi}{n} = 2\pi\sqrt{\frac{m}{k}}$.

We can conclude that the larger the mass, the longer the period, and the stronger the spring (that is, the larger the stiffness constant), the shorter the period.

Links forward

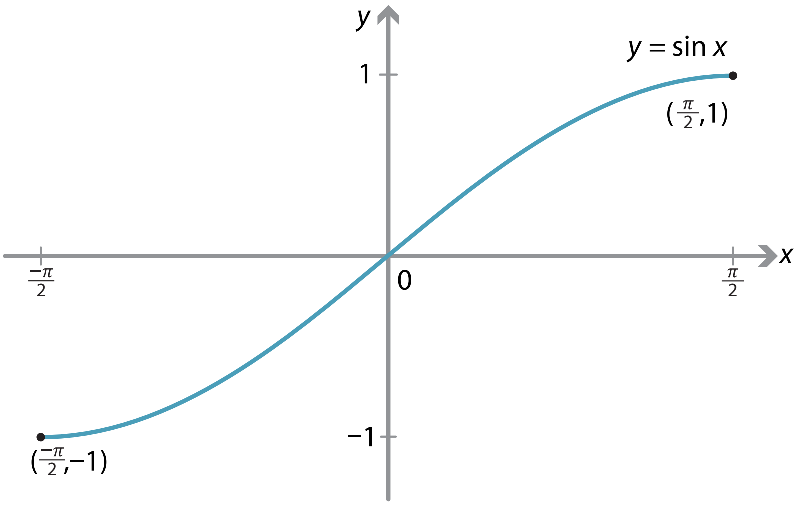
Inverse trigonometric functions

The sine and cosine functions are not one-to-one, and therefore they do not possess inverses. To overcome this problem, we have to restrict their domains, and find inverses for these functions with restricted domains. These issues are discussed in the module *Functions II*.

We will briefly discuss the sine function, and leave cosine as a directed exercise.

Note that $\sin 0 = \sin \pi$, and so our chosen domain cannot include both 0 and π. We can see that sin is a one-to-one function on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, but not on any larger interval containing the origin.

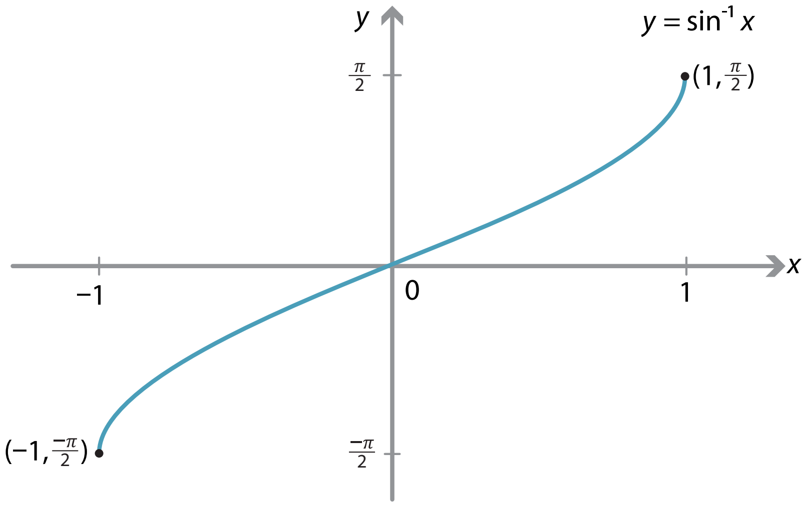
We restrict the domain of $y = \sin x$ to $[-\frac{\pi}{2}, \frac{\pi}{2}]$.



This restricted function, with domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and range $[-1, 1]$, is one-to-one. Hence, it has an inverse function denoted by $f(x) = \sin^{-1} x$, which is read as *inverse sine of* x. (This inverse function is also often denoted by $\arcsin x$.)

It is important not to confuse $\sin^{-1} x$ with $(\sin x)^{-1} = \dfrac{1}{\sin x}$; these are two completely different functions.

The graph of $y = \sin^{-1} x$ is drawn as follows.



The domain of $\sin^{-1}$ is $[-1, 1]$ and its range is $[-\frac{\pi}{2}, \frac{\pi}{2}]$. We can see from the graph that $\sin^{-1}$ is an odd function, that is, $\sin^{-1}(-x) = -\sin^{-1} x$. We can also see that it is an increasing function.

An obvious (and interesting) question to ask is 'What is its derivative?'

The derivative of inverse sine

We recall from the module *Introduction to differential calculus* that $\frac{dy}{dx} \times \frac{dx}{dy} = 1$.

Let $y = \sin^{-1} x$. Then $x = \sin y$, and so

$$\begin{aligned} \frac{dx}{dy} &= \cos y \\ &= \sqrt{1 - \sin^2 y} \qquad (\text{since } \cos y \geq 0) \\ &= \sqrt{1 - x^2}. \end{aligned}$$

Hence

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}.$$

We observe from the graph of $y = \sin^{-1} x$ that the gradient of the curve is positive. We can also see that the gradient approaches infinity as we approach $x = 1$ or $x = -1$.

Using the chain rule, it is easy to show that

$$\frac{d}{dx}\left(\sin^{-1}\frac{x}{a}\right) = \frac{1}{\sqrt{a^2 - x^2}}$$

and so

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a} + C,$$

which gives us a new and important integral.

Exercise 13

a Restrict the domain of $y = \cos x$ to the interval $[0, \pi]$. Draw a sketch to show that this restricted function is one-to-one, and write down the domain and range of its inverse $y = \cos^{-1} x$.

b Explain graphically why $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$, and find $\cos^{-1}(-\frac{1}{2})$.

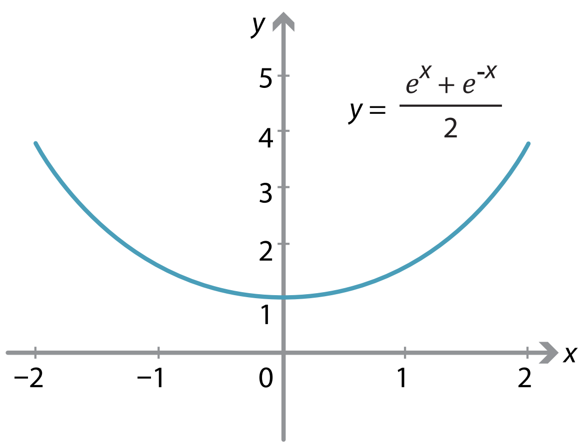
c Show that the derivative of $\cos^{-1} x$ is $\dfrac{-1}{\sqrt{1 - x^2}}$.

d Let $f(x) = \sin^{-1} x + \cos^{-1} x$, for $x \in [-1, 1]$. Find the derivative of f, and conclude that f is a constant function and find its value.

The hyperbolic functions

In the 17th century, the mathematician Johann Bernoulli (and others) studied the curve produced by a hanging chain. It had been (incorrectly) thought by some that the curve was a parabola, but Bernoulli showed that its equation is very different. The curve is often referred to as a **catenary** (from the Latin word for *chain*) and, with appropriate choice of origin and scale, has the equation

$$C(x) = \frac{e^x + e^{-x}}{2}.$$



The catenary curve.

If we define

$$S(x) = \frac{e^x - e^{-x}}{2},$$

then it is easy to see that $S'(x) = C(x)$ and $C'(x) = S(x)$. Moreover, $C(0) = 1$ and $S(0) = 0$, and so these functions bear some similarity to the trigonometric functions cos and sin — although of course they are not periodic. (They do, however, have a complex period.) Hence, by analogy, $C(x)$ is written as $\cosh x$ (pronounced *cosh of x*) and $S(x)$ is written as $\sinh x$ (usually pronounced *shine of x*).

These function have other similarities to the trigonometric functions. For example, analogous to $\cos^2 x + \sin^2 x = 1$, we have

$$\cosh^2 x - \sinh^2 x = 1.$$

There are many other interesting analogues.

These functions are often called the **hyperbolic functions** (hence the 'h'), because this last identity enables us to parameterise half of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

by $x = a\cosh t$, $y = b\sinh t$.

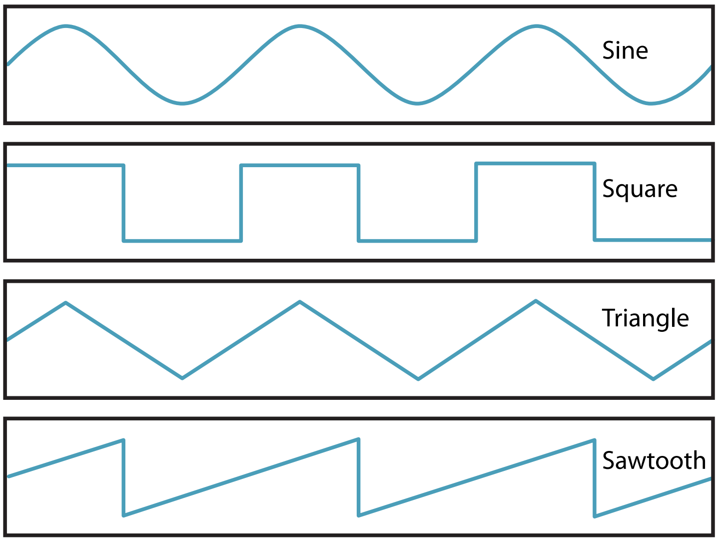
The set of functions consisting of polynomials, rational functions, exponential and logarithmic functions, trigonometric functions and hyperbolic functions is often referred to as the set of **elementary functions**, since they are the most commonly occurring and well studied of all functions. Other functions which are not combinations of the above are sometimes referred to as **special functions**. An example of a special function is $f(x) = \int_0^x e^{-t^2}\,dt$.

History and applications

Some of the history of trigonometry was covered in earlier modules. The famous mathematician Euler, and his contemporaries Jacob and Johann Bernoulli, applied the ideas of calculus to the trigonometric functions producing remarkable results.

The mathematician Joseph Fourier, after whom **Fourier series** are named, applied the ideas of trigonometric series to solve physical problems — in particular, he looked at the distribution of heat in a metal bar.

One of the main modern applications of the trigonometric functions is to the analysis of signals and waves. Many different types of waves arise in the study of alternating currents and signals.



All of our modern telecommunications and electronic devices were only made possible by an understanding of the physics of electricity and the modern development of electrical engineering.

To give just a little insight into this, we are going to show how to build a 'sawtooth' wave using trigonometric functions.

The basic identities

$$\cos A \cos B = \frac{1}{2}\big(\cos(A-B) + \cos(A+B)\big)$$
$$\sin A \sin B = \frac{1}{2}\big(\cos(A-B) - \cos(A+B)\big)$$
$$\sin A \cos B = \frac{1}{2}\big(\sin(A-B) + \sin(A+B)\big)$$

can be established by expanding the right-hand sides. Using these identities, we can easily prove that, if m, n are positive integers, then

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0 & \text{if } m \neq n, \\ \pi & \text{if } m = n. \end{cases}$$

A similar formula holds for $\cos mx \cos nx$. We can also show that

$$\int_{-\pi}^{\pi} \cos mx \sin nx \, dx = 0,$$

for any integers m, n.

We will attempt to construct the function $y = x$, for $x \in [-\pi, \pi]$, using an infinite sum of sine functions. We choose sine because the function $y = x$ is odd, and so is the sine function. Thus we write:

$$\begin{aligned} x &= b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots \\ &= \sum_{n=1}^{\infty} b_n \sin nx. \end{aligned}$$

Multiplying both sides by $\sin x$ and integrating from $-\pi$ to π, we have

$$\begin{aligned} \int_{-\pi}^{\pi} x \sin x \, dx &= b_1 \int_{-\pi}^{\pi} \sin x \sin x \, dx + b_2 \int_{-\pi}^{\pi} \sin x \sin 2x \, dx + \cdots \\ &= b_1 \pi, \end{aligned}$$

so

$$b_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin x \, dx.$$

Using the same idea, we can see that

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx \, dx.$$

This integral can be found using a technique known as *integration by parts*, giving

$$b_n = -\frac{2}{n} \cos n\pi.$$

The first few coefficients are

$$b_1 = 2, \quad b_2 = -1, \quad b_3 = \frac{2}{3}, \quad \ldots$$

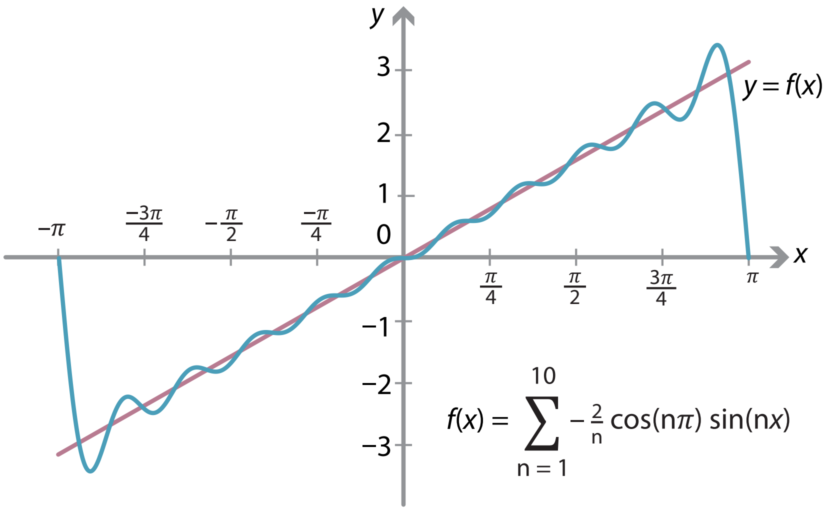
and so we have

$$x = 2\Big(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \frac{1}{5}\sin 5x - \frac{1}{6}\sin 6x + \cdots\Big),$$

at least for $-\pi < x < \pi$.

It needs to be mentioned here that there are serious issues to be examined in regard to convergence, since we have an infinite series, but these issues are beyond the scope of this module.

Graphically, we can chop the series off after a finite number of terms and plot. The following graph shows the sum of the first 10 terms.



Since the trigonometric functions are periodic, the series is converging to a periodic function, which takes copies of the line $y = x$ and repeats them every 2π. This is an approximation of the sawtooth function shown at the start of this section. Note what happens near and at the jump discontinuity!

Answers to exercises

Exercise 1

$$\lim_{x\to 0}\frac{\tan x}{x} = \lim_{x\to 0}\left(\frac{\sin x}{x}\times\frac{1}{\cos x}\right) = 1\times 1 = 1.$$

Exercise 2

$$\lim_{x\to 0}\frac{\sin 3x+\sin 7x}{5x} = \lim_{x\to 0}\left(\frac{3}{5}\times\frac{\sin 3x}{3x}\right)+\lim_{x\to 0}\left(\frac{7}{5}\times\frac{\sin 7x}{7x}\right) = \frac{3}{5}+\frac{7}{5} = 2.$$

Exercise 3

a Multiplying top and bottom by $1+\cos x$ gives

$$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{\sin^2 x}{x^2(1+\cos x)} = \left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2\times\left(\lim_{x\to 0}\frac{1}{1+\cos x}\right) = \frac{1}{2}.$$

b Hence, for small x, we have

$$\frac{1-\cos x}{x^2}\approx\frac{1}{2},$$

and so $\cos x \approx 1-\frac{1}{2}x^2$.

Exercise 4

a We use two compound-angle formulas:

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B.$$

Let $C = A+B$ and $D = A-B$. Then $A = \frac{1}{2}(C+D)$ and $B = \frac{1}{2}(C-D)$. So it follows that

$$\cos C - \cos D = -2\sin A \sin B = -2\sin\Big(\frac{C+D}{2}\Big)\sin\Big(\frac{C-D}{2}\Big).$$

b

$$\begin{aligned}\frac{d}{dx}(\cos x) &= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}\\ &= \lim_{h\to 0}\frac{-2\sin(x+\frac{h}{2})\sin\frac{h}{2}}{h}\\ &= -\Big(\lim_{h\to 0}\sin\big(x+\frac{h}{2}\big)\Big)\times\Big(\lim_{h\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\Big) = -\sin x.\end{aligned}$$

Exercise 5

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\Big(\frac{\sin x}{\cos x}\Big) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

Exercise 6

a $\displaystyle\frac{d}{dx}(\operatorname{cosec} x) = \frac{d}{dx}(\sin x)^{-1} = -1(\sin x)^{-2}\times\cos x = -\frac{\cos x}{\sin^2 x} = -\operatorname{cosec} x\cot x.$

b $\displaystyle\frac{d}{dx}(\sec x) = \frac{d}{dx}(\cos x)^{-1} = -1(\cos x)^{-2}\times -\sin x = \frac{\sin x}{\cos^2 x} = \tan x\sec x.$

c $\displaystyle\frac{d}{dx}(\cot x) = \frac{d}{dx}\Big(\frac{\cos x}{\sin x}\Big) = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\operatorname{cosec}^2 x.$

Exercise 7

$$\begin{aligned}\frac{d}{dx}\log_e\Big(\frac{1+\sin x}{\cos x}\Big) &= \frac{d}{dx}\big(\log_e(1+\sin x) - \log_e(\cos x)\big)\\ &= \frac{\cos x}{1+\sin x} + \frac{\sin x}{\cos x} = \frac{\cos^2 x + \sin^2 x + \sin x}{(1+\sin x)\cos x}\\ &= \frac{1+\sin x}{(1+\sin x)\cos x} = \frac{1}{\cos x} = \sec x.\end{aligned}$$

Exercise 8

The derivative of the function is

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\sin x}{3+4\cos x}\right) = \frac{3\cos x+4}{(3+4\cos x)^2}.$$

Since $\cos x \geq -1$, it follows that $\dfrac{dy}{dx} > 0$ whenever $3+4\cos x \neq 0$. Hence, y is an increasing function wherever it is defined.

Exercise 9

The perimeter of the triangle is $P = 2a+2a\cos\theta = 2a(1+\cos\theta)$, so

$$a = \frac{P}{2(1+\cos\theta)}.$$

The area of the triangle is

$$A = \frac{1}{2}\big(2a\cos\theta\big)\big(a\sin\theta\big) = a^2\cos\theta\,\sin\theta = \frac{1}{2}a^2\sin 2\theta.$$

Substituting for a gives

$$A = \frac{P^2\sin 2\theta}{8(1+\cos\theta)^2}.$$

We want to find θ in the range 0 to $\frac{\pi}{2}$ such that $\dfrac{dA}{d\theta} = 0$. By the quotient rule, if $\dfrac{dA}{d\theta} = 0$, then

$$16P^2\cos 2\theta\,(1+\cos\theta)^2 + 16P^2\sin 2\theta\,(1+\cos\theta)\,\sin\theta = 0.$$

It follows after some calculation that $(1+\cos\theta)^2(2\cos\theta-1) = 0$. So $\cos\theta = -1$ or $\cos\theta = \frac{1}{2}$. Hence, for $0 < \theta < \frac{\pi}{2}$, the only solution is $\theta = \frac{\pi}{3}$. The triangle is equilateral.

Exercise 10

a $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (\sin 2x+\cos 3x)\,dx = \Big[-\frac{1}{2}\cos 2x+\frac{1}{3}\sin 3x\Big]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{1}{6}.$

b We have $\dfrac{d}{dx}(x\sin x) = \sin x + x\cos x$. Hence

$$\int_0^{\frac{\pi}{2}} x\cos x\,dx = \Big[x\sin x\Big]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x\,dx = \frac{\pi}{2}-1.$$

c $\displaystyle\int \tan^2 x\,dx = \int \big(\sec^2 x - 1\big)\,dx = \tan x - x + C.$

Exercise 11

We have $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$, and so

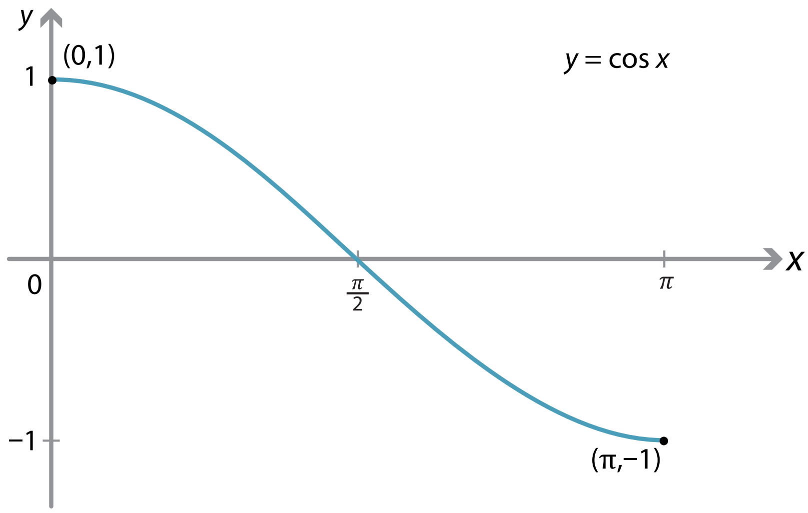
$$\int \sin^2\theta \, d\theta = \frac{1}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C.$$

Exercise 12

Since the period is $\pi\sqrt{2}$, we have $\dfrac{2\pi}{n} = \pi\sqrt{2}$, so $n = \sqrt{2}$. The amplitude is $C = 2$. Hence, from $v^2 = n^2(C^2 - x^2)$, when $x = 0$, $v = \pm 2\sqrt{2}$. Thus the speed is $2\sqrt{2}$ m/s.

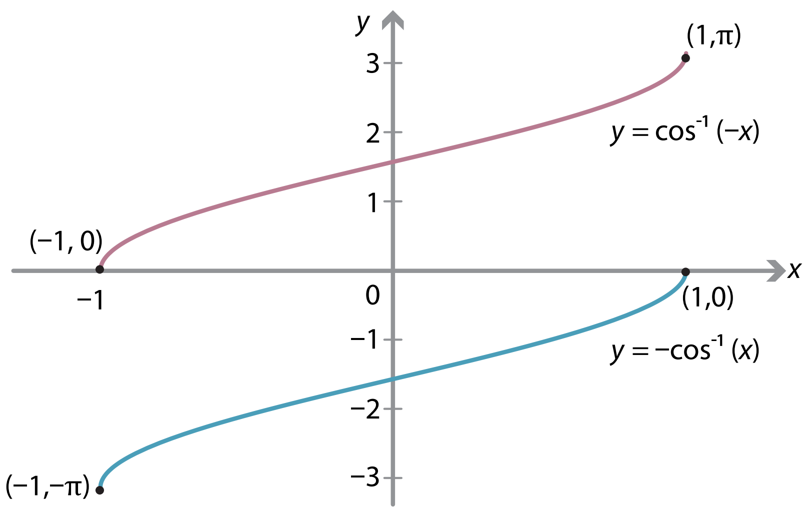
Exercise 13

a



This restricted function has domain $[0, \pi]$ and range $[-1, 1]$. So its inverse has domain $[-1, 1]$ and range $[0, \pi]$.

b



Since $\cos\frac{\pi}{3} = \frac{1}{2}$, we have $\cos^{-1}(\frac{1}{2}) = \frac{\pi}{3}$ and so $\cos^{-1}(-\frac{1}{2}) = \pi - \cos^{-1}(\frac{1}{2}) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$.

c Let $y = \cos^{-1} x$. Then $x = \cos y$, and so

$$\frac{dx}{dy} = -\sin y = -\sqrt{1-\cos^2 y} = -\sqrt{1-x^2}.$$

(Note that $0 \le y \le \pi$ and so $0 \le \sin y \le 1$.) Hence,

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}.$$

d Let $f(x) = \sin^{-1} x + \cos^{-1} x$, for $x \in [-1, 1]$. Then

$$f'(x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0.$$

So $f(x) = C$, for some constant C. Now $f(0) = \frac{\pi}{2}$ and so $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$.

0 1 2 3 4 5 6 7 8 9 10 11 12