University of Babylon
College of Engineering
Mechanical Engineering Dept.

Subject : Mathematics III
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LINES AND PLANES IN SPACE

Equation of lines in space

1) **Vector form**:

Let (x_0, y_0, z_0) and (x_1, y_1, z_1) be points of a line L in space and let (x, y, z) be any other point on the line. Consider the vectors $\vec{u} = \langle x_0, y_0, z_0 \rangle$ and $\vec{v} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$ and $\vec{w} = \langle x, y, z \rangle$. The equation of the line L is given by $\vec{w} = \vec{u} + t\vec{v}$ where t ε real numbers $\vec{v}$ is called the direction vector of the line.

2) **Parametric form of the equation of a line**.

Let {a, b, c} be a set of direction numbers for the line L, where $a = x_1 - x_0$, $b = y_1 - y_0$ and $c = z_1 - z_0$. For any point (x, y, z) on the line

$$\begin{cases} x = x_o + t\,(x_1 - x_o) \\ y = y_o + t\,(y_1 - y_o) \\ z = z_o + t\,(z_1 - z_o) \end{cases} \quad \text{or} \quad \begin{cases} x = x_o + at \\ y = y_o + at \\ z = z_o + at \end{cases} \quad \text{where t is a scalar}$$

3) **Symmetric form: solving for t we obtain**

$$\frac{x - x_o}{a} = \frac{y = y_o}{b} = \frac{z - z_0}{c}, \quad provideda \neq o, b \neq o \; and \; c \neq 0$$

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Two lines in space:

Two lines in space are

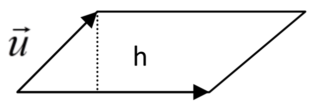
1) Coincident if they are the same line

2) Parallel if they have no points in common but belong to the same plane.

3) Intersecting if they have only one point in common.

4) Skew lines if they have no points in common and belong to different planes.

Area of a parallelogram: Area= $|\vec{u} \times \vec{v}|$



Proof :

$\mathbf{A = bh}$, but $\sin\theta = \dfrac{\mathbf{h}}{|\vec{\mathbf{u}}|} \rightarrow \mathbf{h} = |\vec{\mathbf{u}}|\sin\theta$

Therefore, $\mathbf{A} = |\vec{\mathbf{v}}||\vec{\mathbf{u}}|\sin\theta = |\vec{\mathbf{u}} \times \vec{\mathbf{v}}|$

Area of a triangle = $\dfrac{1}{2}|\vec{\mathbf{u}} \times \vec{\mathbf{v}}|$

The proof follows from the area of a parallelogram because the area of a triangle is half the area of the parallelogram.

Problem.

Find the area of the triangle ABC in space determined by the point A=(2, -3, 5),

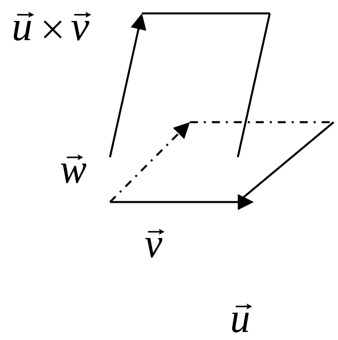
B=(-2, 1, 4) and C=(4, -1, -2)

Solution:

$$\vec{\mathbf{u}} = \mathbf{A\vec{B}} = \langle -4, 4, -1\rangle \text{ and } \vec{\mathbf{v}} = \mathbf{A\vec{C}} = \langle 2, 2, -7\rangle,\ \vec{\mathbf{u}} \times \vec{\mathbf{v}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 4 & -1 \\ 2 & 2 & -7 \end{vmatrix} = -26\mathbf{i} - 30\mathbf{j} - 16\mathbf{k}$$

Area= $(1/2)\sqrt{(-26)^2 + (-30)^2 + (-16)^2} = (1/2)\sqrt{1832} = \sqrt{458} \approx 21.4$

Volume of a parallelepiped $\left|\vec{w}\bullet\left(\vec{u}\times\vec{v}\right)\right|$



Volume=(area of the base)(height)

$$h=\left|Proj_{\vec{u}\times\vec{v}}\vec{w}\right|=\frac{\left|\vec{w}\bullet\left(\vec{u}\times\vec{v}\right)\right|}{\left|\vec{u}\times\vec{v}\right|}$$

$$Volume=\left|\vec{u}\times\vec{v}\right|\frac{\left(\vec{w}\bullet\left(\vec{u}\times\vec{v}\right)\right)}{\left|\vec{u}\times\vec{v}\right|}=\left|\vec{w}\bullet\left(\vec{u}\times\vec{v}\right)\right|$$

Planes in space: any equation of the form

Vector representation of a plane:

If (x_0, y_0, z_0) is a point in a plane with normal vector $\langle A, B, C\rangle$, then any point (x, y, z) of the plane satisfies the vector equation

$$\langle A,B,C\rangle\bullet\langle x-x_0, y-y_0, z-z_0\rangle=0$$

Parametric representation of a plane:

If (x_0, y_0, z_0) is a point in a plane, and $\langle a_1, b_1, c_1\rangle$ *and* $\langle a_2, b_2, c_2\rangle$ are vectors in the plane, then any point (x, y, z) can be represented parametrically by the system of equation

$$\begin{cases} x= & a_1r+a_2s & +x_0 \\ y= & b_1r+b_2s & +y_0 \quad \text{where r and s are scalar numbers} \\ z= & c_1r+c_2s & +z_0 \end{cases}$$

Rectangular representation of a plane:

Ax + By + Cz + D =0 or a(x - x_0) + B(y - y_0) + C(z - z_0) = 0 where (x_0, y_0, z_0) is a given point of the plane and the vector Ai +Bj + Ck is normal (perpendicular) vector to the plane.

How to determine the equation of a plane

1) Find two non-parallel vectors $\vec{u}$ and $\vec{v}$ on the plane.

2) $\vec{u} \times \vec{v}$ determines a vector perpendicular or normal to the plane.

The coordinates of such vector determine A, B, C.

3) Use one of the points to determine D.

How to determine the equation of a plane given three noncollinear points P, Q, R

1) Determine vectors $\vec{u}$ from P to Q and $\vec{v}$ from P to R (or R to Q)

2) $\vec{u} \times \vec{v}$ determines a normal vector (A, B, C) to the plane.

3) Use the normal vector and any of the points to find the plane.

Parallel planes:

two planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ are parallel

if there exist a real number k such that $A_1 = kA_2$, $B_1 = kB_2$, $C_1 = kC_2$, $D_1 = kD_2$.

The normal vectors of two parallel planes are normal

Perpendicular planes:

If wo planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ are perpendicular then $A_1A_2 + B_1B_2 + C_1C_2 = 0$ (the dot product of their normal lines is zero)

How to find the distance between a point and a plane:

Let $Ax + By + Cz + D = 0$ and (x_1, y_1, z_1) be a point.

Let $\vec{u} = \langle x - x_1,\ y - y_1\ z - z_1 \rangle$ be a vector from the given point to a point on the line.

The vector $\vec{v} = \langle A,\ B,\ C \rangle$ is normal to the plane.

The desired distance is given by

$$\left|Proj_{\vec{v}}\vec{u}\right| = \frac{|\vec{u} \bullet \vec{v}|}{|\vec{v}|} = \frac{|A(x - x_1) + B(y - y_1) + C(z - z_1)|}{\sqrt{A^2 + B^2 + C^2}}$$

$$= \frac{|Ax + By + Cz - Ax_1 - By_1 - Cz_1|}{\sqrt{A^2 + B^2 + C^2}} = \frac{|-D - Ax_1 - By_1 - Cz_1|}{\sqrt{A^2 + B^2 + C^2}}$$

$$\textbf{Distance} = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

Note: to find the distance between two parallel planes in space, take any point of one of the planes and find the distance between the point and the other plane.

Formula for the distance between a line and a point in R^2

By a similar argument for the distance between a plane and a point in R^2,

If follows that

$$\textbf{Distance} = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

Note: to find the distance between two parallel lines in the plane, take any point of one of the lines and find the distance between the point and the other line.

Line normal to a plane:

The line $\frac{x - x_1}{A} = \frac{y - y_1}{B} = \frac{z - z_1}{C}$ is perpendicular

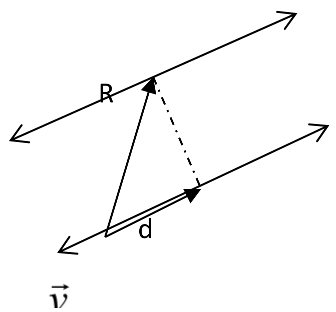
to the plane Ax + By +Cz +D =0, provided A≠0, B≠0and C≠0.

How to find the distance between two lines in space?

1. If the lines are coincident then d=0

2. If the lines intersect at one point, d=0.

3. **Distance between two parallel lines**. If the lines are parallel, $d = \frac{|\vec{u} \times \vec{v}|}{|\vec{u}|}$ where $|\vec{u}|$ is a vector in the direction of the line and v is a vector from one point on the line to a point on the other line. This formula can also be used to find the distance from a point to a line.



Let θ be the angle between $\vec{u}$ and $\vec{v}$

$$\sin\theta = \frac{d}{|\vec{v}|} \rightarrow d = |\vec{v}|\sin\theta. \text{ Therefore } d = \frac{|\vec{u}||\vec{v}|\sin\theta}{|\vec{u}|} = \frac{|\vec{u} \times \vec{v}|}{|\vec{u}|}$$

4. **Distance between two skewed lines**. If the lines are skew, $d = \frac{|\vec{w} \cdot (\vec{u} \times \vec{v})|}{|\vec{u} \times \vec{v}|}$ where $\vec{u}$ is a direction of L_1, $\vec{v}$ is a direction of L_2, $\vec{w}$ is a vector containing a point of each line.

Proof: it follows from the fact that $\vec{u} \times \vec{v}$ is perpendicular to both $\vec{u}$ *and* $\vec{v}$ and therefore

The distance $d = \mathrm{Proj}_{\vec{u} \times \vec{v}} \vec{w} = \frac{|\vec{w} \bullet (\vec{u} \times \vec{v})|}{|\vec{u} \times \vec{v}|}$