NORMAL SUBGROUP

Definition 2 -11.

A subgroup (H ,\* ) of the group (G , \* ) is said to be . NORMAL SUBGROUP ( or INVARIANT ) in ( G , \* ) if . and only if every left coset of H in G is also a right coset . of H in G .

Definition 2\_11/ .

A subgroup(H,\*)is normal subgroup of the group(G ,\*)if . and only if a \*H = H \* a for every a ε G .

Examples .

If H={ f1 (x) = x , f4(x)=(x-1) / x, f6 (x) =1/(1-x)} , show that . (H , o ) is normal .

If K = { f1 (x) = x , f2 (x) = 1/ x } , show that (K , o ) is not . normal .

Exercise.

Find all normal subgroups of S3 and Z10 .

Theorem 2 -15 .

The subgroup (H ,\*) is normal subgroup of the group . (G , \*)If and only if for each a G,

a \* H \* a-1  ⊆H

Proof .

Suppose that H is normal , we must prove that . a \* H \* a-1 ⊆H.

Let a\* h1 \* a-1 a\* H \* a-1.

Since a \* H = H \* a

Then there exists h2 H such that

a\* h1 = h2 \* a .

a\* h1 \*a-1 = h2

Hence a\* H \*a-1 ⊆H . 34

Conversely , suppose a\* H \*a-1 ⊆H , we must prove that

a\* H = H \* a.

Let a \* h a\* H.

Since a\* H \*a-1 ⊆ H ,

Then a\* h\* a-1 = h1 for some h1 ε H .

thus

a \* h = (a \* h \* a-1 ) \* a = h1\* a .

but h1 \* a H \* a .

then a \* H ⊆H \* a .

Similarlly H \* a a \* H .

Then a \* H = H \* a .

Exercise .

Show that ( cent G , \* ) is a normal subgroup of each . group ( G , \* ) .

Proof .

Let a G .

a\* cent G = { a \* c | c cent G }

= { c \* a | c cent G }

= cent G \* a .

Then cent G is normal .

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QUOTINRT GROUP

Definition 2 -12 .

If ( H , \* ) is a normal subgroup of the group ( G , \* ) . . then we shall denoted the collection of distant coset of H . in G by G/ H :

G / H = { a \* H │ a ε G }.

We will denoted the binary operation on G /H by 

and define as :

(a\* H )( b \* H ) = ( a \* b ) \* H .

Exercise .

Show that ( G / H ,) is group .

Proof.

First we must prove that ⊗ is well define .

Suppose that

a \* H = a1 \* H and b \*H = b1 \* H .

hence, a-1 \* a1 , b-1 \* b1H

Since H is normal in G , then

x \* H \* x-1 ⊆ H for every x in G .

Then,

b-1 \* H \* b = b-1 \* H \* (b-1)-1⊆ H ,

so b-1 \* (a-1 \* a1 ) \* b H

but

(a \* b)-1 \* (a1 \* b1) = ( b-1 \* (a-1 \* a1) \* b) \* ( b-1 \* b1) H ?

hence

( a \* b ) \* H = ( a1 \* b1 ) \* H .

Then ⊗ is well – defined .

It is clear that G / H ≠ ᶲ?

G/H is closed under ⊗?

⊗ is associative ?

H is identity of G/H ?

(a \* H )-1 = a-1 \* H ?

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Definition 2 -13 .

If ( H , \* ) is normal subgroup of the group ( G , \* ) , . then (G/H , ⊗) is called QUOTIENT GROUP of G by H .

Example 1.

If S = { r360 , r180 } , fined G/S .

Example 2.

The cyclic subgroup ((n), +) is normal of (Z , +),where . n is nonnegative integer . If

a + (n) = {a + kn |kZ }= [a].

( a + (n))⊗ (b + (n)) = a + b + (n).

[a] ⊗ [b] = [a+b].

Hence ( Z/ (n) ,⊗ ) = (Zn , +n )

Definition 2-14.

Given a group (G,\*) and elements a,b G, the commutator

of a and b is denoted by [ a , b]defined to be the product . a \* b \* a-1 \* b-1 .

Notes:

1)If G is commutative then [a , b ] =e .

2)a \* b= [a , b ] \*b \*a .

3)[a , b ]-1  =[b , a ] .

4)[ G ,G ] = { 𝞟 [ai , bi ] | ai , biG } is called derived . subgroup or commutator subgroup .

Theorem 2 – 16 .

The group ([G,G] ,\* ) is a normal subgroup of ( G , \*) .

Proof .

Let aGand c [G ,G ],

a\* c \* a-1 = (a \* c \*a-1\*c-1) \* c

=[a , c ] \* c [G,G] ?

Hence [G , G ] is normal subgroup of G .

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Theorem 2 – 17 .

Let (H, \* ) be a normal subgroup of the group ( G , \* ) .

The quotint group ( G/H , ⊗ ) is commutative if and only . if [ G , G ] ⊆H .

Proof .

Suppose that a \* H and b \* H G/H.

Since H is the identity element of G/H ,

Then ⊗ is commutative ⟺ H = [ a \* H , b \* H ]

= (a \* H )⊗ (b \* H ) ⊗ ( a \* H )-1⊗ ( b \* H ) -1

= ( a \* b \* a-1 \* b -1) \* H

⟺ [a , b ] H .

⟺ [ G , G ] ⊆ H .

Corollary .

For any group ( G , \* ) the commutator quotint group

(G / [G , G ] , ⊗ ) is commutative .

Proof .

Exercise .

Defintion 2 - 15 .

The group G is called SIMPLE if and only if no triveal

normal subgroup of G .

Exercises .

1. Prove that the intersaction of two normal subgroups is also normal subgroup .
2. Let ((a) , \* ) be cyclic group of order 15 . Fined ((a) /(a3), ⊗ ) .
3. Show that a group ( G , \* ) is commutative if and only if [ G , G ] = {e } .

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