NORMAL SUBGROUP

 Definition 2 -11.

 A subgroup (H ,\* ) of the group (G , \* ) is said to be . NORMAL SUBGROUP ( or INVARIANT ) in ( G , \* ) if . and only if every left coset of H in G is also a right coset . of H in G .

 Definition 2\_11/ .

 A subgroup(H,\*)is normal subgroup of the group(G ,\*)if . and only if a \*H = H \* a for every a ε G .

 Examples .

 If H={ f1 (x) = x , f4(x)=(x-1) / x, f6 (x) =1/(1-x)} , show that . (H , o ) is normal .

 If K = { f1 (x) = x , f2 (x) = 1/ x } , show that (K , o ) is not . normal .

 Exercise.

 Find all normal subgroups of S3 and Z10 .

 Theorem 2 -15 .

 The subgroup (H ,\*) is normal subgroup of the group . (G , \*)If and only if for each a $\in $ G,

 a \* H \* a-1  ⊆H

 Proof .

 Suppose that H is normal , we must prove that . a \* H \* a-1 ⊆H.

 Let a\* h1 \* a-1 $\in $ a\* H \* a-1.

 Since a \* H = H \* a

 Then there exists h2 $\in $ H such that

 a\* h1 = h2 \* a .

 a\* h1 \*a-1 = h2

 Hence a\* H \*a-1 ⊆H . 34

 Conversely , suppose a\* H \*a-1 ⊆H , we must prove that

 a\* H = H \* a.

 Let a \* h $\in $ a\* H.

 Since a\* H \*a-1 ⊆ H ,

 Then a\* h\* a-1 = h1 for some h1 ε H .

 thus

 a \* h = (a \* h \* a-1 ) \* a = h1\* a .

 but h1 \* a $\in $ H \* a .

 then a \* H ⊆H \* a .

 Similarlly H \* a a \* H .

 Then a \* H = H \* a .

 Exercise .

 Show that ( cent G , \* ) is a normal subgroup of each . group ( G , \* ) .

 Proof .

 Let a$\in $ G .

 a\* cent G = { a \* c | c$\in $ cent G }

 = { c \* a | c $\in $ cent G }

 = cent G \* a .

 Then cent G is normal .

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 QUOTINRT GROUP

 Definition 2 -12 .

 If ( H , \* ) is a normal subgroup of the group ( G , \* ) . . then we shall denoted the collection of distant coset of H . in G by G/ H :

 G / H = { a \* H │ a ε G }.

 We will denoted the binary operation on G /H by 

 and define as :

 (a\* H )( b \* H ) = ( a \* b ) \* H .

 Exercise .

 Show that ( G / H ,) is group .

 Proof.

 First we must prove that ⊗ is well define .

 Suppose that

 a \* H = a1 \* H and b \*H = b1 \* H .

 hence, a-1 \* a1 , b-1 \* b1$\in $H

 Since H is normal in G , then

 x \* H \* x-1 ⊆ H for every x in G .

 Then,

 b-1 \* H \* b = b-1 \* H \* (b-1)-1⊆ H ,

 so b-1 \* (a-1 \* a1 ) \* b $\in $H

 but

 (a \* b)-1 \* (a1 \* b1) = ( b-1 \* (a-1 \* a1) \* b) \* ( b-1 \* b1) $\in $ H ?

 hence

 ( a \* b ) \* H = ( a1 \* b1 ) \* H .

 Then ⊗ is well – defined .

 It is clear that G / H ≠ ᶲ?

 G/H is closed under ⊗?

 ⊗ is associative ?

 H is identity of G/H ?

 (a \* H )-1 = a-1 \* H ?

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 Definition 2 -13 .

 If ( H , \* ) is normal subgroup of the group ( G , \* ) , . then (G/H , ⊗) is called QUOTIENT GROUP of G by H .

 Example 1.

 If S = { r360 , r180 } , fined G/S .

 Example 2.

 The cyclic subgroup ((n), +) is normal of (Z , +),where . n is nonnegative integer . If

 a + (n) = {a + kn |k$\in $Z }= [a].

 ( a + (n))⊗ (b + (n)) = a + b + (n).

 [a] ⊗ [b] = [a+b].

 Hence ( Z/ (n) ,⊗ ) = (Zn , +n )

 Definition 2-14.

 Given a group (G,\*) and elements a,b$\in $ G, the commutator

 of a and b is denoted by [ a , b]defined to be the product . a \* b \* a-1 \* b-1 .

 Notes:

 1)If G is commutative then [a , b ] =e .

 2)a \* b= [a , b ] \*b \*a .

 3)[a , b ]-1  =[b , a ] .

 4)[ G ,G ] = { 𝞟 [ai , bi ] | ai , bi$\in $G } is called derived . subgroup or commutator subgroup .

 Theorem 2 – 16 .

 The group ([G,G] ,\* ) is a normal subgroup of ( G , \*) .

 Proof .

 Let a$\in $Gand c $\in $[G ,G ],

 a\* c \* a-1 = (a \* c \*a-1\*c-1) \* c

 =[a , c ] \* c $\in $[G,G] ?

 Hence [G , G ] is normal subgroup of G .

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 Theorem 2 – 17 .

 Let (H, \* ) be a normal subgroup of the group ( G , \* ) .

 The quotint group ( G/H , ⊗ ) is commutative if and only . if [ G , G ] ⊆H .

 Proof .

 Suppose that a \* H and b \* H $\in $ G/H.

 Since H is the identity element of G/H ,

 Then ⊗ is commutative ⟺ H = [ a \* H , b \* H ]

 = (a \* H )⊗ (b \* H ) ⊗ ( a \* H )-1⊗ ( b \* H ) -1

 = ( a \* b \* a-1 \* b -1) \* H

 ⟺ [a , b ] $\in $ H .

 ⟺ [ G , G ] ⊆ H .

 Corollary .

 For any group ( G , \* ) the commutator quotint group

 (G / [G , G ] , ⊗ ) is commutative .

 Proof .

 Exercise .

 Defintion 2 - 15 .

 The group G is called SIMPLE if and only if $∄$ no triveal

 normal subgroup of G .

 Exercises .

1. Prove that the intersaction of two normal subgroups is also normal subgroup .
2. Let ((a) , \* ) be cyclic group of order 15 . Fined ((a) /(a3), ⊗ ) .
3. Show that a group ( G , \* ) is commutative if and only if [ G , G ] = {e } .

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