3.5 Phase and group velocities

We know matter waves (de Broglie waves)

$$\lambda = \frac{h}{p} = \frac{h}{m\mathrm{v}}$$

where v is the velocity of a particle of the mass *m*.

For particles and according to relativity

$$E = mc^2 \text{ and since } E = h\nu \text{ then}$$

$$mc^2 = h\nu \rightarrow \nu = \frac{mc^2}{h}$$

If we define u as the phase velocity and is given by

$$u = \lambda\nu = \frac{h}{m\mathrm{v}}\frac{mc^2}{h} = \frac{c^2}{\mathrm{v}}$$

Thus, the velocity of the matter waves is larger than speed of light in vacuum, u > c, unless v > c.

This result seems disturbing because it appears that the matter waves would propagate faster than the speed of light and would not be able to keep up with particles whose motion they govern.

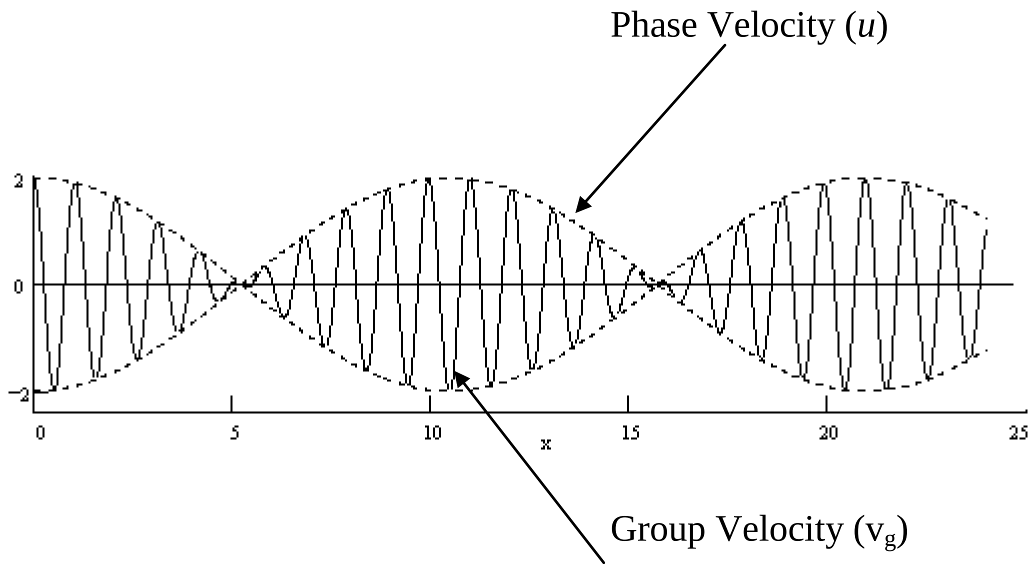
However, the phase velocity of a wave is the velocity of the wave front, not its amplitude. The maximum of the amplitude of a given wave can propagate at different velocity, called group velocity. At this velocity the energy (information) is transmitted. Usually, v_g = u, but in the case of dispersion, u(ν), the group velocity v_g < u. Thus,

the matter wave should be dispersive to match the requirement of $v_g < u$ when $u > c$.

The group velocity is given by

$$v_g = \frac{d\omega}{d\kappa}$$

where ω is the angular frequency and κ is the wave number.



since $\omega = 2\pi\nu$ **and** $\kappa = \frac{2\pi}{\lambda}$ **then we get**

$$u = \lambda\nu = \frac{2\pi}{\kappa}\frac{\omega}{2\pi} = \frac{\omega}{\kappa} \Rightarrow u = \frac{\omega}{\kappa}$$

Question:

The dispersion relation for free relativistic electron waves is

$$\omega = \sqrt{c^2\kappa^2 + \left(mc^2/\hbar\right)^2}$$

(a) Calculate expressions for the phase velocity u and group velocity v_g of these waves and show that their product is constant, independent of κ.

(b) From the result (a), what can you conclude about v_g if u > c?

Solution:

(a) From the definition of the phase velocity

$$u = \frac{\omega}{\kappa} = \sqrt{c^2 + \left(\frac{mc^2}{\kappa\hbar}\right)^2}$$

We see that the phase velocity u > c.

From the definition of the group velocity

$$v_g = \frac{d\omega}{d\kappa} = \frac{1}{2}\frac{2c^2\kappa}{\sqrt{c^2\kappa^2 + \left(mc^2/h\right)^2}}$$

$$= \frac{c^2\kappa}{c\kappa\sqrt{1+\left(mc/\kappa\hbar\right)^2}} = \frac{c}{\sqrt{1+\left(mc/\kappa\hbar\right)^2}}$$

Thus, the group velocity v_g < c.

However, the product

$$u v_g = \sqrt{c^2 + \left(\frac{mc^2}{\kappa\hbar}\right)^2} \times \frac{c}{\sqrt{1+\left(mc/\kappa\hbar\right)^2}} = c^2$$

is constant independent of κ.

(b) We see from (a) that in general for dispersive waves for which *u* > c, the group velocity v_g < c. Only when *u* = *c*, the group velocity v_g = c .