Riemann Sum Practice Problems

Questions:

1. Approximate the area under the curve
   \[ f(x) = x^2 + 2, \quad -2 \leq x \leq 1 \]
   with a Riemann sum, using:
   
   a) Three sub-intervals and left endpoints.
   b) Six sub-intervals and right endpoints.

2. Approximate the area under the curve
   \[ f(x) = \sqrt{x+1}, \quad -1 \leq x \leq 0 \]
   with a Riemann sum, using:
   
   a) Four sub-intervals and left endpoints.
   b) Three sub-intervals and midpoints.

3. Approximate the value of the integral:
   \[ \int_{0}^{3} 6 - 2x \, dx \]
   with a Riemann sum, using:
   
   a) Six sub-intervals and left endpoints.
   b) Six sub-intervals and right endpoints.
   c) Sketch the graph of \( y = 6 - 2x \), and the blocks representing the Riemann sums in a), b).
   Compute the integral using geometry.
   
   d) If \( f(x) \) is a decreasing function, what can we say about the value of the Riemann sums using left endpoints as sample points? What about right endpoints?
   e) If \( f(x) \) is an increasing function, what can we say about the value of the Riemann sums using left endpoints as sample points? What about right endpoints?
4. Approximate the value of the integral:

\[
\int_1^3 3 - x^3 \, dx
\]

with a Riemann sum, using:

a) Three sub-intervals and right endpoints.

b) Using five sub-intervals and left endpoints.

c) Compute the exact value of the integral using the Fundamental Theorem of Calculus.

5. A rectangular canal, 5m wide and 100m long has an uneven bottom. Depth measurements are taken at every 20m along the length of the canal. Use these depth measurements to construct a Riemann sum using right endpoints to estimate the volume of water in the canal.

<table>
<thead>
<tr>
<th>Distance</th>
<th>0m</th>
<th>20m</th>
<th>40m</th>
<th>60m</th>
<th>80m</th>
<th>100m</th>
</tr>
</thead>
<tbody>
<tr>
<td>Depth</td>
<td>2.0m</td>
<td>1.6m</td>
<td>1.8m</td>
<td>2.1m</td>
<td>2.1m</td>
<td>1.9m</td>
</tr>
</tbody>
</table>
Solutions:

1. a) \( a = -2, \ b = 1, \ \Delta x = \frac{b-a}{n} = \frac{1-(-2)}{3} = 1, \ f(x) = x^2 + 2 \)

\[
\text{Area} = \left( \sum f(x_i)\Delta x \right) = \left( f(-2) + f(-1) + f(0) \right)\Delta x
\]
\[
= \left[ (-2)^2 + 2 \right] + \left[ (-1)^2 + 2 \right] + \left[ (0)^2 + 2 \right]
\]
\[
= 11
\]

b) \( a = -2, \ b = 1, \ \Delta x = \frac{b-a}{n} = \frac{1-(-2)}{6} = \frac{1}{2}, \ f(x) = x^2 + 2 \)

\[
\text{Area} = \left( \sum f(x_i)\Delta x \right) = \left( f(-1.5) + f(-1) + f(-0.5) + f(0) + f(0.5) + f(1) \right)\Delta x
\]
\[
= \left[ (-1.5)^2 + 2 \right] + \left[ (-1)^2 + 2 \right] + \left[ (-0.5)^2 + 2 \right] + \left[ (0)^2 + 2 \right] + \left[ (0.5)^2 + 2 \right] + \left[ (1)^2 + 2 \right] \frac{1}{2}
\]
\[
= 8.375
\]

2. a) \( a = -1, \ b = 0, \ \Delta x = \frac{b-a}{n} = \frac{0-(-1)}{4} = \frac{1}{4}, \ f(x) = \sqrt{x+1} \)

\[
\text{Area} = \left( \sum f(x_i)\Delta x \right) = \left( f(-1) + f(-0.75) + f(-0.5) + f(-0.25) \right)\Delta x
\]
\[
= \left( \sqrt{0} + \sqrt{0.25} + \sqrt{0.5} + \sqrt{0.75} \right) 0.25 = 0.5183
\]

b) \( a = -1, \ b = 0, \ \Delta x = \frac{b-a}{3} = \frac{0-(-1)}{3} = \frac{1}{3}, \ f(x) = \sqrt{x+1} \)

Remember: mid-points are \( \Delta x / 2 \) from the endpoints, but still \( \Delta x \) apart.

\[
\text{Area} = \left( \sum f(x_i)\Delta x \right) = \left( f\left( \frac{5}{6} \right) + f\left( -\frac{3}{6} \right) + f\left( -\frac{1}{6} \right) \right)\Delta x
\]
\[
= \left( \sqrt{\frac{1}{6}} + \sqrt{\frac{1}{2}} + \sqrt{\frac{5}{6}} \right) \frac{1}{3} = 0.6761
\]
3. a) \(a = 0, \ b = 3, \ \Delta x = \frac{b-a}{n} = \frac{3-0}{6} = \frac{1}{2}, \ f(x) = 6 - 2x\)

\[
\text{Area} = \left( \sum_{i=1}^{n} f(x_i) \Delta x \right) = (f(0) + f(0.5) + f(1) + f(1.5) + f(2) + f(2.5)) \Delta x \\
= ([6-2(0)] + [6-2(0.5)] + [6-2(1)] + [6-2(1.5)] + [6-2(2)] + [6-2(2.5)])0.5 \\
= ([6] + [5] + [4] + [3] + [2] + [1])0.5 = 21/2 \\
= 10.5
\]

b) \(a = 0, \ b = 3, \ \Delta x = \frac{b-a}{n} = \frac{3-0}{6} = \frac{1}{2}, \ f(x) = 6 - 2x\)

\[
\text{Area} = \left( \sum_{i=1}^{n} f(x_i) \Delta x \right) = (f(0.5) + f(1) + f(1.5) + f(2) + f(2.5) + f(3)) \Delta x \\
= ([6-2(0.5)] + [6-2(1)] + [6-2(1.5)] + [6-2(2)] + [6-2(2.5)] + [6-2(3)])0.5 \\
= ([5] + [4] + [3] + [2] + [1] + [0])0.5 = 15/2 \\
= 7.5
\]

c) \[\text{Left Endpoint Sample Points} \quad \text{Right Endpoint Sample Points} \quad \text{Area} = 0.5(\text{base})(\text{height}) = 0.5(3)(6) = 9\]

d) As in the example above, a decreasing function means that left sample points will make boxes higher than our graph, and thus overestimate our integral. Right endpoints, similarly lie under our graph, and underestimate our integral.

e) If the graph is increasing, the situation is reversed. Left sample points will produce boxes below the graph, and thus underestimate the integral. Right sample points will produce boxes above the graph, and overestimate the integral.
4.a) $a=1, \ b=3, \ \Delta x=\frac{3-1}{3}=\frac{2}{3}, \ f(x)=3-x^3$

$$\text{Area} = \left( \sum f(x_i)\Delta x \right) = \left( f\left(\frac{5}{3}\right) + f\left(\frac{7}{3}\right) + f\left(3\right) \right)\left(\frac{2}{3}\right)$$

$$= \left( 3-\left(\frac{5}{3}\right)^3 \right) + \left( 3-\left(\frac{7}{3}\right)^3 \right) + \left( 3-(3)^3 \right) \left(\frac{2}{3}\right) = -23.556$$

b) As before, $a=1, \ b=3, \ f(x)=3-x^3, \ \text{but} \ \Delta x=\frac{3-1}{5}=\frac{2}{5}$

$$\text{Area} = \left( \sum f(x_i)\Delta x \right) = \left( f\left(1\right) + f\left(\frac{7}{5}\right) + f\left(\frac{9}{5}\right) + f\left(\frac{11}{5}\right) + f\left(\frac{13}{5}\right) \right)$$

$$= \left( 3-(1)^3 \right) + \left( 3-\left(\frac{7}{5}\right)^3 \right) + \left( 3-\left(\frac{9}{5}\right)^3 \right) + \left( 3-\left(\frac{11}{5}\right)^3 \right) + \left( 3-\left(\frac{13}{5}\right)^3 \right) \left(\frac{2}{5}\right)$$

$$= -9.120$$

c) \[ \int_1^3 3-x^3 \, dx = 3x-\frac{1}{4}x^4 \bigg|_1^3 = \left( 9-\frac{81}{4} \right) - \left( 3-\frac{1}{4} \right) = 6-20 = -14 \]

5. Volume = (width)(cross section area)

$$= (5\text{m}) \left( \sum f(x_i)\Delta x \right) = (5\text{m}) \left( f(20) + f(40) + f(60) + f(80) + f(100) \right) \Delta x$$

$\Delta x$ here is the distance between our measurements, 20m, so we get:

$$\text{Volume} = (5)(20) \left( f(20) + f(40) + f(60) + f(80) + f(100) \right)$$

$$= 100(1.6+1.8+2.1+2.1+1.9)$$

$$= 950\text{m}^3$$