On some types of $\lambda$-continuous function in Bitopological Spaces

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Abstract:
[singal and singal ,[ 1968] introduced several properties of almost continuous mapping and [T.Noiri , 1990] introduced a weak form of faint continuity . [Hassna H. and sajda K. , 2005] introduced $\lambda$-continuity in bitopological spaces . the purpose of this paper is to introduce and study some types of $\lambda$-continuity in bitopological spaces with some relation between them.

Introduction:
The open set In the special bitopological space $(X\Psi,\Psi^\alpha)$ denoted by $\lambda$-open set and the collection of all $\lambda$-open sets forms a topological space on $X$ denoted by $\Psi$, greater than $\Psi$ . [N.Levine , 1961] introduced the concept of weak continuity as a generalized of continuity , later [Hussain ,1966] introduced almost continuity as another generalization and [Anderew and whitlesy , 1966]introduced the concept of closure continuity which is stronger than weak continuity .[singal and singal , 1968] introduced anew almost continuity which is different from that of hussain.the purpose of this paper is to further the study of the concept of strong $\lambda$-continuity and almost strongly $\lambda$-continuity ,faintly $\lambda$-continuity ,and weakly $\lambda$-continuity in bitopological spaces.

A function $f$: $(X,\Psi,\Psi^\alpha)$→$(Y,\xi,\xi^\alpha)$ is weakly $\lambda$-continuous function at a point $x\in X$ if given any $\xi$-open set $V$ in $Y$ containing $f(x)$ ,there exist $\lambda$-open set $U$ containing $x$ such that $f(U)\subseteq cl_{\xi}(V)$.if the condition is satisfied at each $x\in X$ then $f$ is said to be weakly continuous function. the function $f$ is strongly $\lambda$-continuous function at $x\in X$ if given any $\xi$-open set $V$ in $Y$ containing $f(x)$ ,there exist $\lambda$-open set $U$ containing $x$ such that $f(cl_{\Psi}(U))\subseteq(V)$.if the condition is satisfied at each $x\in X$ then $f$ is said to be strongly $\lambda$-continuous function, and it is called almost $\lambda$-continuous if for each point $x\in X$ and each $\xi$-open set $V$ in $Y$ containing $f(x)$ there exist $\lambda$-open set $U$ in $X$ containing $x$ such that $f(U)\subseteq cl_{\xi}(V)$ .the function $f$ is said to be almost strongly $\lambda$-continuous if and only if for ach $x\in X$ and $\xi$-nbd $V$ of $f(x)$ there exist $\lambda$-open set $U$ containing $x$ such that $f(cl_{\Psi}(U))\subseteq int_{\xi}(cl_{\xi}(V))$. A bitopological space $(X,\Psi,\Psi^\alpha)$ is called urysohn if for every $x\neq y$ in $X$ there exist $\Psi$-open sets $U,V$ in $X$ such that $x\in U$ and $y\in V$ and $cl_{\Psi}(U)\cap cl_{\Psi}(V)=\emptyset$ and if $(X,\Psi)$ is Urysohn space then $(X, \Psi,\Psi^\alpha)$ is also urysohn space..

From the definitions stated above ,one can easily obtain the following diagram

Strongly $\lambda$-continuous $\Rightarrow$ $\lambda$-continuous $\Leftrightarrow$ Faintly-continuous

↓

Almost strongly $\lambda$-continuous $\Rightarrow$ almost $\lambda$-continuous $\Rightarrow$ weakly $\lambda$-continuous
1- \( \lambda \)-continuous and strongly \( \lambda \)-continuous function

**Theorem (1-1):** let \( f : (X, \Psi, \Psi^\alpha_0) \rightarrow (Y, \xi, \xi^\alpha_0) \) be strongly \( \lambda \)-continuous and \( g : (Y, \xi, \xi^\alpha_0) \rightarrow (Z, \omega, \omega^\alpha_0) \) be \( \lambda \)-continuous function then \( gof \) is strongly \( \lambda \)-continuous.

**Proof:** exist by definition.

**Theorem (1-2):** if \( f : (X, \Psi, \Psi^\alpha_0) \rightarrow (Y, \xi, \xi^\alpha_0) \) is \( \lambda \)-continuous such that \( X \) is extremely disconnected then \( f \) is strongly \( \lambda \)-continuous.

**Proof:** let \( x \in X \) and \( V \) be \( \xi \)-nbd of \( f(x) \) in \( Y \), since \( f \) is \( \lambda \)-continuous there exist \( \lambda \)-open set \( U \) containing \( x \) such that \( f(U) \subseteq V \) and since \( (X, \Psi) \) is extremely disconnected then \( f(\text{cl}_\Psi(U)) \subseteq V \) then \( f \) is strongly \( \lambda \)-continuous.

**Theorem (1-3):** A function \( f : (X, \Psi, \Psi^\alpha_0) \rightarrow (Y, \xi, \xi^\alpha_0) \) is \( \lambda \)-continuous if and only if \( \text{cl}_\Psi(f^{-1}(V)) \subseteq f^{-1}(\text{cl}_\xi(V)) \) for each \( \xi \)-open set \( V \) in \( Y \).

**Proof:** let \( V \) be \( \xi \)-open set in \( Y \), then \( f^{-1}(\text{cl}_\xi(V)) \) is \( \lambda \)-closed set in \( X \), now \( V \subseteq \text{cl}_\xi(V) \) then \( f^{-1}(V) \subseteq f^{-1}(\text{cl}_\xi(V)) \) and then \( \text{cl}_\Psi(f^{-1}(V)) \subseteq \text{cl}_\Psi(f^{-1}(\text{cl}_\xi(V))) = f^{-1}(\text{cl}_\xi(V)) \).

Conversely, let \( V \) be \( \xi \)-closed set in \( Y \) then \( \text{cl}_\Psi(f^{-1}(V)) \subseteq f^{-1}(\text{cl}_\xi(V)) = f^{-1}(V) \). now since \( f^{-1}(V) \subseteq \text{cl}_\Psi(f^{-1}(V)) \) then \( \text{cl}_\Psi(f^{-1}(V)) = f^{-1}(V) \) for \( f^{-1}(V) \) is \( \lambda \)-closed set. \( f \) is \( \lambda \)-continuous function.

**Theorem (1-4):** let \( f : (X, \Psi, \Psi^\alpha_0) \rightarrow (Y, \xi, \xi^\alpha_0) \) be strongly \( \lambda \)-continuous such that \( X \) is regular space and \( g : (X, \Psi, \Psi^\alpha_0) \rightarrow (X \times Y, \Psi \times \xi, \Psi^\alpha \times \xi^\alpha) \) be the graph mapping given by \( g(x) = (x, f(x)) \) for each \( x \in X \) then \( g \) is strongly \( \lambda \)-continuous.

**Proof:** let \( x \in X \) and let \( w \) be \( \Psi \times \xi \)-open set in \( X \times Y \) containing \( g(x) \). then there exist \( \Psi \)-open set \( G \) in \( X \) and \( \xi \)-open set \( H \) in \( Y \) such that \( g(x) = (x, f(x)) \subseteq G \times H \), since \( X \) is regular space then there exist \( \Psi \)-open set \( B \) in \( X \) such that \( \text{cl}_\Psi(B) \subseteq G \). since \( f \) is strongly \( \lambda \)-continuous there exist \( \lambda \)-open set \( U \) containing \( x \) such that \( U \subseteq G \) and \( f(U) \subseteq H \). there for \( g(U) \subseteq \text{cl}_\Psi \times \text{cl}_\xi(H) \subseteq G \times H \subseteq w \) then \( g \) is \( \lambda \)-continuous.

**Theorem (1-5):** let \( (X, \Psi, \Psi^\alpha_0) \rightarrow (Y, \xi, \xi^\alpha_0) \) be \( \lambda \)-continuous and \( g : (X, \Psi, \Psi^\alpha_0) \rightarrow (X \times Y, \Psi \times \xi, \Psi^\alpha \times \xi^\alpha) \) be the graph mapping given by \( g(x) = (x, f(x)) \) for each \( x \in X \) then \( g \) is \( \lambda \)-continuous.

**Proof:** let \( x \in X \) and let \( w \) be \( \Psi \times \xi \)-open set in \( X \times Y \) containing \( g(x) \) then there exist \( \Psi \)-open set \( G \) in \( X \) and \( \xi \)-open set \( H \) in \( Y \) such that \( g(x) = (x, f(x)) \subseteq G \times H \subseteq w \), since \( f \) is \( \lambda \)-continuous there exist \( \lambda \)-open set \( U \) containing \( x \) such that \( U \subseteq G \) and \( f(U) \subseteq H \). there for \( g(U) \subseteq G \times H \subseteq w \) then \( g \) is \( \lambda \)-continuous.

**Definition (1-6):** let \( f : (X, \Psi, \Psi^\alpha_0) \rightarrow (X, \Psi_A, \Psi^\alpha_A) \) be a function such that \( X \) is Urysohn space where \( A \subseteq X \) and \( f|A \) is the identity function on \( A \) then \( f \) is \( \lambda \)-retraction

**Theorem (1-7):** let \( A \) be a subset of \( X \) and \( f : (X, \Psi, \Psi^\alpha_0) \rightarrow (X, \Psi_A, \Psi^\alpha_A) \) be \( \lambda \)-retraction of \( X \) onto \( A \).\( f \) is \( \lambda \)-retraction if \( X \) is Urysohn space then \( A \) is \( \lambda \)-closed subset of \( X \).

**Proof:** suppose that there exist \( x \in \text{cl}_\Psi(A) / A \) since \( f \) is \( \lambda \)-retraction we have \( f(x) \neq x \) . since \( X \) is Urysohn space then there exist \( \lambda \)-open sets \( U, V \) such that \( x \in U \) and \( f(x) \in V \), \( \text{cl}_\Psi(U) \cap \text{cl}_\Psi(V) = \emptyset \). Let \( w \) be any \( \lambda \)-open set containing \( x \) then \( U \cap w \) is \( \lambda \)-open sets containing \( x \) and then \( \text{cl}_\Psi(U \cap w) \cap A = \emptyset \). Now since \( x \in \text{cl}_\Psi(A) \) then \( y \in \text{cl}_\Psi(U \cap w) \cap A \). since \( y \in A \), \( f(y) = y \in \text{cl}_\Psi(U) \) and hence \( f(y) \notin \text{cl}_\Psi(V) \), from that we get \( f(\text{cl}_\Psi(w)) \) is not contained in \( \text{cl}_\Psi(V) \) and this is contradiction.
2-$\lambda$-continuous function and weakly $\lambda$-continuous.

**Theorem (2-1):** If let $f: (X, \Psi, \Psi^\alpha) \rightarrow (Y, \xi, \xi^\alpha)$ is weakly $\lambda$-continuous such that $(X, \Psi)$ extremely disconnected then $\text{cl}_\Psi(f^{-1}(V)) \subseteq f^{-1}(\text{cl}_\xi(V))$ for each $\xi$-open set $V$ in $Y$.

*Proof:* Let $V$ be $\xi$-open in $Y$, then $f^{-1}(\text{cl}_\xi(V))$ is $\lambda$-open in $X$ and $f^{-1}(V) \subseteq f^{-1}(\text{cl}_\xi(V))$, then $\text{cl}_\Psi(f^{-1}(V)) \subseteq \text{cl}_\Psi(f^{-1}(\text{cl}_\xi(V)))$, since $X$ is extremely disconnected $\text{cl}_\Psi(f^{-1}(V)) \subseteq f^{-1}(\text{cl}_\xi(V))$.

**Theorem (2-2):** A mapping $f: (X, \Psi, \Psi^\alpha) \rightarrow (Y, \xi, \xi^\alpha)$ is weakly $\lambda$-continuous if and only if $f^{-1}(V) \subseteq \text{int}_\Psi(f^{-1}(\text{cl}_\xi(V)))$.

*Proof:* Let $f$ is weakly $\lambda$-continuous and $V$ be $\xi$-open set in $Y$ then $f^{-1}(\text{cl}_\xi(V))$ is $\lambda$-open in $X$, since $\text{int}_\Psi(f^{-1}(\text{cl}_\xi(V))) = f^{-1}(\text{cl}_\xi(V))$ then $f^{-1}(V) \subseteq f^{-1}(\text{cl}_\xi(V)) = \text{int}_\Psi(f^{-1}(\text{cl}_\xi(V)))$.

Sufficiency, let $x \in X$ and $V$ be $\xi$-nbhd of $f(x)$ in $Y$ then $x \in f^{-1}(V) \subseteq \text{int}_\Psi(f^{-1}(\text{cl}_\xi(V)))$ let $U = \text{int}_\Psi(f^{-1}(\text{cl}_\xi(V)))$ then $U$ is $\lambda$-open set containing $x$ and $f(U) \subseteq \text{cl}_\xi(V)$ and there for $f$ is weakly $\lambda$-continuous.

**Theorem (2-3):** Let $f: (X, \Psi, \Psi^\alpha) \rightarrow (Y, \xi, \xi^\alpha)$ is weakly $\lambda$-continuous such that $(Y, \xi)$ is extremely disconnected space then $f$ is $\lambda$-continuous.

*Proof:* Let $x \in X$ and $V$ be $\xi$-nbhd of $f(x)$, since $f$ is weakly $\lambda$-continuous there exist $\lambda$-open set $U$ containing $x$ such that $f(U) \subseteq \text{cl}_\xi(V)$, since $(Y, \xi)$ is extremely disconnected then $f(U) \subseteq V$ and then $f$ is $\lambda$-continuous.

**Theorem (2-4):** Let $f: (X, \Psi, \Psi^\alpha) \rightarrow (Y, \xi, \xi^\alpha)$ is weakly $\lambda$-continuous such that $(Y, \xi)$ is extremely disconnected space then $f$ is almost $\lambda$-continuous.

*Proof:* See the proof of theorem (2-3)

III- $\lambda$-continuous function and almost $\lambda$-continuous.

**Theorem (3-1):** If the mapping $f: (X, \Psi, \Psi^\alpha) \rightarrow (Y, \xi, \xi^\alpha)$ is $\lambda$-continuous such that $(X, \Psi)$ is extremely disconnected then $\text{cl}_\Psi(f^{-1}(V)) \subseteq f^{-1}(\text{cl}_\xi(V))$ for each $\xi$-open set $V$ in $Y$.

*Proof:* Let $V$ be $\xi$-open set in $Y$ and since $f$ is $\lambda$-continuous $f^{-1}(V)$ is $\lambda$-open in $X$, since we have that $f^{-1}(V) \subseteq f^{-1}(\text{cl}_\xi(V))$ and $(X, \Psi)$ is extremely disconnected, there for $\text{cl}_\Psi(f^{-1}(V)) \subseteq f^{-1}(\text{cl}_\xi(V))$.

**Theorem (3-2):** Let $f: (X, \Psi, \Psi^\alpha) \rightarrow (Y, \xi, \xi^\alpha)$ is almost $\lambda$-continuous and $(Y, \xi)$ extremely disconnected then $f$ is $\lambda$-continuous function.

*Proof:* Let $x \in X$ and $V$ be $\xi$-nbhd of $f(x)$, since $f$ is almost $\lambda$-continuous there exist $\lambda$-open set $U$ containing $x$ such that $f(U) \subseteq \text{cl}_\xi(V)$, since $f(U) \subseteq \text{cl}_\xi(V)$ then $f(U) \subseteq \text{cl}_\xi(V)$.

**Theorem (3-3):** Let $f: (X, \Psi, \Psi^\alpha) \rightarrow (Y, \xi, \xi^\alpha)$ is almost $\lambda$-continuous $f^{-1}(V)$ $\lambda$-pen set in $X$ for each regular -open set $V$ (resp. semi- open set $V$) in $Y$.

*Proof:* Exist by definition.

**Proposition (3-5):** A function $f: (X, \Psi, \Psi^\alpha) \rightarrow (Y, \xi, \xi^\alpha)$ is almost $\lambda$-continuous if and only if $f^{-1}(V) \subseteq \text{int}_\Psi(f^{-1}(\text{int}_\xi(V)))$ for each $\xi$-open set $V$ in $Y$.

*Proof:* Let $f$ is almost $\lambda$-continuous and $V$ is $\xi$-open set in $Y$ then $f^{-1}(\text{int}_\xi(V))$ is $\lambda$-pen set in $X$ and $V \subseteq \text{int}_\xi(\text{cl}_\xi(V))$ then we have that $f^{-1}(V) \subseteq f^{-1}(\text{int}_\xi(\text{cl}_\xi(V))) = \text{int}_\Psi(f^{-1}(\text{int}_\xi(\text{cl}_\xi(V)))$.

Sufficiency, let $x \in X$ and $V$ be $\xi$-nbhd of $f(x)$ in $Y$ then
\[ x \in f^1(V) \subseteq \text{int}_\xi(f^{-1}(\text{int}_\xi(\text{cl}_\xi(V)))) \text{. let } U = \text{int}_\xi(f^{-1}(\text{int}_\xi(\text{cl}_\xi(V)))) \text{ then } f(U) \subseteq (\text{int}_\xi(\text{cl}_\xi(V)) \text{ there for } f \text{ is almost } \lambda \text{-continuous} .

**Proposition(3-6):** let \( f: (X,\Psi,\Psi^\alpha) \rightarrow (Y,\xi,\xi^\alpha) \) is almost \( \lambda \)-continuous then \( f^1(V) \) is \( \lambda \)-open set in \( X \) for each \( \xi \)-regular open set in \( Y \).

**Theorem(3-7):** almost \( \lambda \)-continuous function \( f: (X,\Psi,\Psi^\alpha) \rightarrow (Y,\xi,\xi^\alpha) \) such that \( Y \) is \( \xi \)-extremely disconnected is \( \lambda \)-continuous if and only if \( \text{int}_\xi(f^{-1}(\text{cl}_\xi(V))) \). let \( U = \text{int}_\xi(f^{-1}(\text{cl}_\xi(V))) \text{ then } U \subseteq f^{-1}(\text{cl}_\xi(V)) \text{ and then } f(U) \subseteq V \) we get \( f^1(V) = f^1(\text{cl}_\xi(V)) \).

**Theorem(3-8) :** let \( f: (X,\Psi,\Psi^\alpha) \rightarrow (Y,\xi,\xi^\alpha) \) is almost \( \lambda \)-continuous such that \( X \) is \( \Psi \)-extremely disconnected then \( f \) is almost strongly \( \lambda \)-continuous.

**Example(3-9):** let \( X = \{a,b,c,d\} \) and \( \Psi = \{X,\varnothing,\{a\},\{a,b\}\} \)
\( \xi = \{X,\varnothing,\{a\}\} \)\( \text{let } f: (X,\Psi,\Psi^\alpha) \rightarrow (Y,\xi,\xi^\alpha) \) defined by \( f(a) = f(b) = p \), \( f(c) = q \) then \( f \) is weakly \( \lambda \)-continuous but it is neither \( \lambda \)-continuous nor almost \( \lambda \)-continuous.

**Example(3-10):** let \( X = \mathbb{R} \) and \( \Psi = \text{co-countable topology} \) and \( Y = \{a,b\} \),
\( \xi = \{Y,\varnothing,\{a\}\} \)\( \text{let } f: (X,\Psi,\Psi^\alpha) \rightarrow (Y,\xi,\xi^\alpha) \) define by \( f(x) = a \) if \( x \in \mathbb{Q} \) and \( f(x) = b \) if \( x \in \mathbb{Q}^c \) then \( f \) is almost \( \lambda \)-continuous and \( \xi \)-continuous but not \( \lambda \)-continuous.

**Definition(3-11):** A function \( f: (X,\Psi,\Psi^\alpha) \rightarrow (Y,\xi,\xi^\alpha) \) is said to be faintly \( \lambda \)-continuous if for each \( x \in X \) and each \( \lambda \)-open set \( V \) containing \( f(x) \) there exist \( \Psi \)-open set \( U \) containing \( x \) such that \( f(U) \subseteq V \).

**Lemma(3-12):** if the function \( f: (X,\Psi,\Psi^\alpha) \rightarrow (Y,\xi,\xi^\alpha) \) is faintly \( \lambda \)-continuous and \( g: (Y,\xi,\xi^\alpha) \rightarrow (Z,\omega,\omega^\alpha) \) is strongly \( \lambda \)-continuous then \( g \circ f \) is strongly \( \lambda \)-continuous.

**Lemma(3-13):** if the function \( f: (X,\Psi,\Psi^\alpha) \rightarrow (Y,\xi,\xi^\alpha) \) is \( \lambda \)-continuous and \( g: (Y,\xi,\xi^\alpha) \rightarrow (Z,\omega,\omega^\alpha) \) is faintly \( \lambda \)-continuous then \( g \circ f \) is \( \lambda \)-continuous.
**Theorem (3-14):** if $f: (X,\Psi,\Psi^\alpha)\to(Y,\xi,\xi^\alpha)$ is faintly $\lambda$-continuous such that $X$ is extremely disconnected then $f$ is strongly $\lambda$-continuous (almost $\lambda$-continuous)

**Proof:** let $x \in X$ and $V$ is $\xi$-open set containing $f(x)$ since $f$ is faintly $\lambda$-continuous and $V$ is $\lambda$-open set in $Y$ there exist $\Psi$-open set $U$ containing $x$ such that $f(U) \subseteq V$, now since $X$ is $\Psi$-extremely disconnected and $U$ is $\lambda$-open set in $X$ $f(cl_\Psi(U)) \subseteq V$ and there for $f$ is strongly $\lambda$-continuous. similar argument uses to proof $f$ is almost $\lambda$-continuous.

**V-Super $\lambda$-continuous and completely $\lambda$-continuous**

**Definition (4-1):** a mapping $f: (X,\Psi,\Psi^\alpha)\to(Y,\xi,\xi^\alpha)$ is super $\lambda$-continuous if and only if for each $x \in X$ and $\xi$-nbd $V$ of $f(x)$ there exist $\lambda$-open set $U$ in $X$ such that $f(int_\Psi(cl_\Psi(U))) \subseteq V$.

**Definition (4-2):** a mapping $f: (X,\Psi,\Psi^\alpha)\to(Y,\xi,\xi^\alpha)$ is completely $\lambda$-continuous if and only if for each $x \in X$ and $\xi$-nbd $V$ of $f(x)$ there exist $\Psi$-regular open set $U$ in $X$ such that $f(U) \subseteq V$.

From these definitions we obtain the following diagram

![Diagram showing relationships between continuous mappings](https://example.com/diagram.png)

**Theorem (4-3):** if the mapping $f: (X,\Psi,\Psi^\alpha)\to(Y,\xi,\xi^\alpha)$ is super $\lambda$-continuous such that $(X,\Psi)$ is semi regular space then $f$ is completely $\lambda$-continuous.

**Proof:** let $x \in X$ and $V$ is $\xi$-nbd of $f(x)$, since $f$ is super $\lambda$-continuous there exist $\lambda$-open set $U$ in $X$ such that $f(int_\Psi(cl_\Psi(U))) \subseteq V$ and since $X$ is $\Psi$-semi regular space then $f(U) \subseteq V$ then $f$ is completely $\lambda$-continuous.

**Theorem (4-4):** if the mapping $f: (X,\Psi,\Psi^\alpha)\to(Y,\xi,\xi^\alpha)$ is completely $\lambda$-continuous such that $(X,\Psi)$ is extremely disconnected then $f$ is almost strongly $\lambda$-continuous.

**Proof:** let $x \in X$ and $V$ is $\xi$-open set containing $f(x)$, since $f$ is completely $\lambda$-continuous there exist regular open set $U$ containing $x$ such that $f(U) \subseteq V$, since $U$ is regular open then $f(U) \subseteq V \subseteq int_\xi(cl_\xi(V))$, also since $X$ is $\Psi$-extremely disconnected then $f(cl_\Psi(U)) \subseteq int_\xi(cl_\xi(V))$.

**Theorem (4-5):** if the mapping $f: (X,\Psi,\Psi^\alpha)\to(Y,\xi,\xi^\alpha)$ is weakly $\lambda$-continuous such that $(X,\Psi)$ is semi regular space and $(Y,\xi)$ is extremely disconnected then $f$ is super $\lambda$-continuous.

**Proof:** let $x \in X$ and $V$ is $\xi$-open set containing $f(x)$, since $f$ is weakly $\lambda$-continuous there exist $\lambda$-open set $U$ containing $x$ such that $f(U) \subseteq cl_\xi(V)$, now since $X$ is $\Psi$-semi regular space and $Y$ is $\xi$-extremely disconnected then $f(int_\Psi(cl_\Psi(U))) \subseteq V$ and then $f$ is super $\lambda$-continuous.

**References:**

