Fuzzy Geometric Distribution with Some Properties

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Abstract

In this Paper ,we drive the fuzzy probability mass function of geometric distribution ,its fuzzy distribution function and some of its properties such as fuzzy mean, like fuzzy variance and fuzzy moment generating function, and we use the fuzzy moment generating function to generate fuzzy moments $\bar{\mathbf{E}}\ (X_{n)}$ for $\mathbf{n} = 1, 2, \ldots$

Keywords : fuzzy probability function , fuzzy probability mass function , fuzzy mean , fuzzy variance , fuzzy moment generating function.

1.1. Introduction

Interest in fuzzy systems in academia, industry, and government is also manifested by the rapid growth of national and international conferences. The continuous probability measure of a *crisp* event, the distribution of which is based by a density *f*(*x*), can be represented by an integral

$$p(A) = \int_A f(x)dx = \int_U \chi_{A(x)} f(x)dx \qquad (1)$$

from which, replacing the characteristic function, characteristic

function $\chi_{A(x)} = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$

by the membership function, immediately the probability of a *fuzzy* event *A* is obtained

$$p(\mathcal{A}) = \int_U \chi_{A(x)} f(x)dx \qquad (2)$$

The analogous construction for *discrete* probabilities is obvious.

This representation was the starting point taken by Zadeh (1968) for defining probabilities for fuzzy sets. The concept proved workable in many applications, although the usual interpretation of probability as the measure of chance for the event that the next realization will fall into the *crisp* set *A* rises considerable difficulties in comprehension: the position of a *crisp* singleton, the realization, within the fuzzy set A would be possible, according to the principle of inclusion for fuzzy sets, only within the *core* A_1 of A. Here an interpretation of $P(A)$ according to **(2)** may help as the *expected value* of the membership function $\mu(X)$

$$P(A) = E_p \mu_A(X) \tag{3}$$

where the index P should point to the distribution, with respect to which the expected value is to be computed; in the example above this would mean the density $f(x)$. In this manner the *statistical* interpretation of this probability becomes possible. Let be $x_1, x_2, \ldots, x_n$ independent realizations of a random variable X with the distribution P and X_i the components of the corresponding sample, then on certain little restrictive conditions, one obtains statements like those known from the laws of large numbers in probability theory

$$P(A) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mu_A(X_i) \tag{4}$$

As a starting point Kruse/Meyer (1987) considered the case that the realizations of a really *crisp* random variable Y can be observed only *fuzzily*. Such an observation is defined as a realization of a *fuzzy coarsening*, a fuzzy random variable Z, from which the characteristic values of the original Y are to be inferredin the set and the fuzzy random variable Z is assumed given. The set of all originals, which can belong to this Z is a fuzzy set over *IY* , which can be assumed a certain type of distributions, then fuzziness can be transferred to obtain a fuzzy set over the parameter domain. By an extension principle all formulae used in probability theory can then be fuzzified. Even if the probabilities $p_i(x_i) \in \underline{P}, 1 \leq i \leq N$, In the considerations on fuzzy probability up to near time, the random events themselves are assumed as *crisp* sets.

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set and let P be a probability function defined on all subsets of X with $p(\{x_i\}) = a_i, 1 \leq i \leq n, 0 \leq a_i \leq 1$ and $\sum_{i=1}^{n} a_i = 1$. That is, X together with P is a discrete (finite) probability distribution. In practice all the a_i values must be known exactly estimated. If an a_i is crisp ,then we will still write it as a fuzzy number even though this fuzzy number crisp. Due to the uncertainty in the a_i values we substitute $\overline{a}_i$, a fuzzy number , and assume that $0 < \overline{a}_i < 1$ for each a_i all i. We will use a fuzzy number for this probability then X together with the a_i values is a discrete (finite) fuzzy probability distribution. We write $\overline{p}$ for fuzzy p and we have $\overline{p}(\{x_i\}) = \overline{a}_i, 1 \leq i \leq n, 0 \leq \overline{a}_i \leq 1$.

In [1],[2] James J . Buckley study the fuzzy Binomial and the fuzzy Poisson.

In this paper we discuss the fuzzy geometric probability mass function, the crisp geometric probability mass function usually written G(p), and drive the fuzzy probability mass function of geometric distribution , its fuzzy distribution function and some of its properties such as fuzzy mean, like fuzzy variance and fuzzy moment generating function, and we use the fuzzy moment generating function to generate fuzzy moments $\overline{\mathrm{E}}(X_n)$ for $n = 1, 2, \ldots$.

1.2. Fuzzy Geometric distribution

Let $X = \{x_1, x_2, \dots, x_n\}$ and let E be anon-empty proper sub set of X . We have an experiment where the result is considered a "success" if the outcome x_i is in E . Otherwise, the result is considered a "failure". Let the probability of success be $p(E) = p$ **for** $0 < p < 1$, and $p(E^c) - q - 1 - p$ be the probability of failure, $0 < p < 1$. A geometric random variable with parameter p has probability mass function [3]

$$P(x) = \begin{cases} pq^{x-1} & x = 1, 2, 3, \dots \\ 0 & otherwise \end{cases} \qquad \textbf{(5)}$$

In this experiment let us assume that $p(E)$ is not known precisely and it need to be estimated, or obtained from expert opinion. So the p values are uncertain and we substitute $\overline{p}$ for p and $\overline{q}$ for **q** . That is a $p \in \overline{p}[1]$ and $q \in \overline{q}[1]$ with $p + q = 1$. Now let $\overline{p}(x)$ be the fuzzy probability of x success in m independent trials of the experiment , then the α-cuts of fuzzy geometric probability is

$$\overline{p}(x)[\alpha] = \{pq^{x-1} | S\} \quad \text{for} \quad 0 \le pq^{x-1} \le 1 \qquad \textbf{(6)}$$

We have now S is the statement "$p \in \overline{p}[\alpha], q \in \overline{q}[\alpha], p + q = 1$"

Notice that : $\overline{p}(x)$ is not $\overline{p}\,\overline{q}^{\,x-1}$

If $\overline{p}(x)[\alpha] = [p_{n_1}(\alpha), p_{n_2}(\alpha)]$ then

$$p_{n_1}(\alpha) = min\{pq^{x-1} | S\} \qquad \textbf{(7)}$$

And **(8)**

$$p_{n_2}(\alpha) = max\{pq^{x-1} | S\}$$

The distribution function for geometric distribution is

$$F(x) = \sum_{k=1}^{x} p(1-p)^{k-1}$$

$$= p\frac{1-(1-p)^x}{1-(1-p)}$$

$$= 1 - (1-p)^x \quad \textbf{for} \quad x = 1, 2, \dots \qquad \textbf{(9)}$$

And the α-cuts of fuzzy geometric distribution is

$$F(x, \overline{p})[\alpha] = \{\textstyle\sum_{k=1}^{x} p(1-p)^{k-1} \mid s\}$$

$$= \left\{p\frac{1-(1-p)^x}{1-(1-p)} \mid s\right\}$$

$$= \{1-(1-p)^x \mid s\} \qquad (10)$$

Then

$$F(x,\overline{p}) - \begin{cases} 1-(1-\overline{p})^x & x-1,2,\dots \\ o & x<1 \end{cases} \qquad (11)$$

Example 1.2.1 : $Let\ p = 0.4\ , q = 0.6$ Since p and **q** are uncertain,

we use $\overline{p} = (0.3/\ 0.4/\ 0.5)$ for **p** and

$\overline{q} = (0.5/\ 0.6/\ 0.7)$ for q .

Now we will calculate the fuzzy number $\overline{p}(2)$

if $p \in \overline{p}[\alpha]$ then $q = 1-p \in \overline{q}[\alpha]$

$$p_{n_1(\alpha)} = min\{pq|s\}$$

$$p_{n_2(\alpha)} = max\{pq|s\}$$

Since $\frac{d(p(1-p))}{dp} > 0$ on $\overline{p}[0]$ we obtain

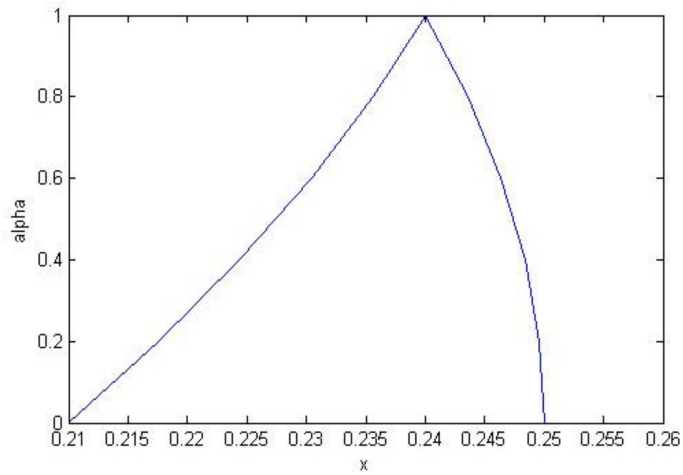
$$\overline{p}[2][\alpha] = [p_1(\alpha)(1-p_1(\alpha)), p_2(\alpha)(1-p_2(\alpha))]$$

Where $\overline{p}[\alpha] = [p_1(\alpha), p_2(\alpha)]$

$$= [0.3+0.1\alpha\ , 0.5-0.1\alpha]$$

Table 1.2.1 : Alpha-cuts of the Fuzzy probability in **Example 1.2.1**

α	$\overline{p}(2)[\alpha]$
0	[0.210 ,0.250]
0.2	[0.217 ,0.249]
0.4	[0.224 ,0.248]
0.6	[0.230 ,0.246]
0.8	[0.235 ,0.243]
1.0	[0.240 ,0.240]



1.3. Fuzzy mean and Fuzzy variance

The α-cuts of the fuzzy mean and the fuzzy variance of the fuzzy geometric distribution are calculated as in the following two equations:

$$\overline{\mu}[\alpha] = \{\sum_{i=1}^{n} x_i a_i \mid S\} \tag{12}$$

where s denotes the statement "$a_i \in \overline{a}_i[\alpha], 1 \leq i \leq n, \sum_{i=1}^{n} a_i$=1"

$$\overline{\sigma}^2[\alpha] = \{\sum_{i=1}^{n}(x_i - \mu)^2 a_i | s, \mu = \sum_{i=1}^{n} x_i a_i\} \tag{13}$$

Theorem 1.3.1:

If a random variable X have fuzzy geometric distribution then

a) $\overline{\mu} = \frac{1}{\overline{p}}$ b) $\overline{\sigma}^2 = 1 - \overline{p}/\overline{p}^2$ c) $m_x\ (t\,, \overline{p}\,) = \frac{\overline{p}e^t}{1\ \ \overline{q}e^t}$

are the fuzzy mean , fuzzy variance and the fuzzy moment generating function respectively.

proof :

a) $\overline{\mu}[\alpha] = \{\sum_{x=1}^{\infty} x(1-p)^{x-1}\, p \mid S\}$

$= \{p \sum_{x=1}^{\infty} x(1-p)^{x-1} \mid S\}$

$= \{p\, p^{-2} \mid S\}$

$= \{p^{-1} | S\}$

$$= \left\{ \frac{1}{p} | S \right\} \tag{14}$$

where $\bar{\mu} = \frac{1}{\bar{p}}$

b) $E\big(X(X-1)\big) = \sum_{x=1}^{\infty} x(x-1)(1-p)^{x-1}\, p$

$$= p(1-p)\sum_{n=1}^{\infty} x(x-1)(1-p)^{x-2}$$

$$= 2p^{-2}(1-p) \tag{15}$$

Now $2p^{-2}(1-p) = E(X^2) - E(X)$

$$2p^{-2}(1-p) = E(X^2) - \left(\frac{1}{p}\right)$$

Then $E(X^2) = 2p^{-2}(1-p) + p^{-1}$

$$= (2-p)p^{-2} \tag{16}$$

$$\sigma^2[\alpha] = \left\{ (2-p)p^{-2} - \left(\frac{1}{p}\right)^2 | S \right\}$$

$$= \{(2-p)p^{-2} - p^{-2} | S\}$$

$$= \{1 - p/p^2 | S\}$$

$$\bar{\sigma}^2 = 1 - \bar{p}/\bar{p}^2 \tag{17}$$

This is the fuzzy variance of fuzzy geometric distribution.

c) Now we will find the fuzzy moment generating function for the fuzzy geometric distribution.

Let $m_x\,(t\,,\bar{p})$ be the fuzzy moment generating function for X and its

α-cuts are determined as $m_x(t,\bar{p})[\alpha] = E(e^{tx})[\alpha]$

$$= \{\textstyle\sum_{x=1}^{\infty} e^{tx}\, q^{x-1} p | S\} \tag{18}$$

$$m_x(t,\bar{p})[\alpha] = \left\{ \frac{pe^t}{1-qe^t} | S \right\}$$

Where $m_x(t,\bar{p}) = \frac{\bar{p}e^t}{1-\bar{q}e^t}$ (19)

Notes that : the fuzzy moment generating function can be used to generate fuzzy moments $\overline{E}(X^n)$ for $n = 1, 2, 3, \ldots$

Using the fuzzy moment generating function we find

$$[dm_x(0,\overline{p})/dt][\alpha] = \{1/p|S\}$$

$$dm_x(0,\overline{p})/dt = 1/\overline{p} \qquad \textbf{(20)}$$

This is the fuzzy mean of the fuzzy geometric distribution. Next we can use the fuzzy moment generating function to get the fuzzy variance of **X**. This computation is a bit more complicated and is done by α-cuts

$$\left[\frac{d^2 m_x(0,\overline{p})}{dt^2} - (dm_x(0,\overline{p})/dt)^2\right][\alpha] = \{1 - p/p^2|S\}$$

$$\frac{d^2 m_x(0,\overline{p})}{dt^2} - (dm_x(0,\overline{p})/dt)^2 = 1 - \overline{p}\,/\overline{p}^{\,2} \qquad \textbf{(21)}$$

This is the fuzzy variance of the fuzzy geometric distribution.

Conclusions

The fuzziness can be transferred to obtain a fuzzy set over the parameter domain. And all formulae used in probability theory can then be fuzzified. So we can obtain the fuzzy geometric(probability, distribution function, mean, variance , and the moment generating function).

Reference

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