The Existence Solution To The Development Heat Equation With Some Conditions

Hayder Jabber Abood and Zainab Hamid Abdali

Department of Mathematics, College of Education for Pure Sciences, Babylon University, Babylon province, Iraq.

Email: drhayder_jabbar@yahoo.com& Email: mentorzainab90@gmail.com

Abstract

We consider the development heat equation with initial boundary conditions. The uniqueness of the solution is hold by using the maximum-minimum principle and some reflection methods.

1.Introduction

In [1], they are studied investigate the inverse problem involving recovery of initial temperature from the information of final temperature profile in a disc, this inverse problem arises when experimental measurements are taken at any given time, and it is desired to calculate the initial profile, they considered the usual heat equation and the hyperbolic heat equation with Bessel operator. In [2] ,they addressed two issues usually encountered when simulating thermal processes in forming processes involving tape-type geometries, as is the case of tape or tow placement, surface treatments, The first issue concerns the necessity of solving the transient model a huge number of times because the thermal loads are moving very fast on the surface of the part and the thermal model is usually non-linear. In [3], they studies coupled heat equations with multi-nonlinearities of six nonlinear Parameters, the critical blow-up exponent is established via a complete classification for all the six nonlinear parameters, where a precise analysis on the geometry of Ω and the absorption coefficients is given for the balanced interaction situation among the multi-nonlinearities, the main attention is contributed to non-simultaneous phenomena in the model to determine the necessary and sufficient conditions of non-simultaneous blow-up with suitable initial data, as well as the conditions under which any blow-up must be non-simultaneous. In [4], they presented a new upper bound of the life span of positive solutions of a semi linear heat equation for initial data having positive limit inferior at space infinity. The upper bound is expressed by the data in limit inferior, not in every direction, but around a specific direction, It is also shown that the minimal time blow-up occurs when initial data attains its maximum at space infinity. In [5], they considered a one-dimensional semi linear parabolic equation $u_t = u_{xx} + e^{ux}$, for which the spatial derivative of solutions becomes unbounded in finite time while the solutions themselves remain bounded, they are established estimate of blowup rate upper and lower bounds, they are proved that in this case the blowup rate does not match the one obtained by the rescaling method. In [6], they considered simultaneous and non-simultaneous blow-up solutions for heat equations coupled via exponential sources, subject to null Dirichlet boundary conditions, the main results complete the previously known results on the optimal classification for simultaneous and non-simultaneous blow-up solutions by covering the whole ranges of exponents, moreover, all kinds of simultaneous and non-simultaneous blow-up rates are obtained.In [7], they are studied the inverse problem of identifying a time dependent unknown coefficient in a parabolic problem subject to initial and non-local boundary conditions along with an over specified condition defined at a specific point in the spatial domain, due to the non-local boundary condition, the system of linear equations resulting from the backward Euler approximation have a coefficient matrix that is a quasi-tridiagonal matrix. In [8], an inverse analysis is performed for simultaneous estimation of relaxation time and order of fractionality in fractional single-phase-lag heat equation, this fractional heat conduction equation is applied on two physical problems, in inverse procedure, solutions of a previously validated linear dual-phase-lag model on the physical problems under study have been used as the measured temperatures, the inverse fractional single-phase-lag heat conduction problem is solved using the nonlinear parameter estimation technique based on the Levenberg–Marquardt method. In [9] they are studied asymptotic behavior in time of small solutions to nonlinear heat equations in subcritical case, they found a new family of self-similar solutions which change a sign. They showed that solutions are stable in the neighborhood of these self-similar solutions. Some results on the construction of asymptotics for solutions of singularly perturbed problem with first-order partial derivatives can be found in [10]. In [11], the authors proposed an algorithm of asymptotic integration of semi-linear initial-boundary-value problems whose minor coefficients are functions oscillating in time with high frequency ω. In [12], the methods proposed in [10] and [11] were combined and an algorithm of asymptotic integration of the initial-boundary-value problem for the heat-conduction equation with nonlinear sources of heat terms oscillating in time with frequency ω^{-1} was developed. Recent studies of asymptotic analysis of differential equations involving large high-frequency terms have been carried out in [13,14]. For a singularly perturbed first-order partial differential equation, a theorem was proved in [15] on the passage to the limit for the case in which the root of the degenerate equation intersect and the root intersection line meets the initial segment on which the initial condition is posed. In [16], the authors considered second-order ordinary differential equation whose coefficients contain smooth and rapidly oscillating summands proportional to the positive powers of the oscillation frequency. A singularly perturbed system of two second-order differential equations(one rapid and one slow), was considered in [17], which proved the existence of a solution and obtained its asymptotics for the case in which the degenerate equation has two intersecting roots. Recent studies of asymptotic analysis of differential equations involving large high-frequency terms have been carried out in [18,19]. Our principal in the present paper we are study the uniqueness of the solution is hold by using the maximum-minimum principle and some reflection methods for the development equation with some conditions.

2. Formulation of the problem

We consider the second-order partial differential equation for the following problem

$$\frac{\partial u(x,t)}{\partial t} - a(x,t)\frac{\partial^2 u(x,t)}{\partial x^2} = 0, \qquad (x,t)\in \Omega = (0<x<l)\times(0<t<T), \tag{2.1}$$

$$u(x,0) = f(x), \qquad 0 \le x \le l,$$

$$u(0,t) = T_1,\ u(l,t) = T_2, \qquad t>0,$$

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = \left.\frac{\partial u(x,t)}{\partial x}\right|_{x=l} = 0, \qquad t>0.$$

where $u(x,t)$ the function of the bar at the point x at the time t, $a(x,t)$ be a continuous function depends on the variable x and t, $f(x)$ is a given function. In order to determine the temperature in the bar at any time t. However it turns out that suffices to consider the case $T_1 = T_2 = 0$ only. We can also assume that the ends of the bar are insulated so that no heat can pass through them, which implies

$$\frac{\partial u(x,t)}{\partial t} - a(x,t)\frac{\partial^2 u(x,t)}{\partial x^2} = 0, \qquad 0 < x < l, t > 0, \tag{2.2}$$

where $u(x,t)$ satisfies the initial condition

$$u(x,0) = f(x), \qquad 0 < x < l, \tag{2.3}$$

And the boundary conditions

$$u(0,t) = u(l,t) = 0, \qquad t > 0, \tag{2.4}$$

In the same way, we can obtain

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = \left.\frac{\partial u(x,t)}{\partial x}\right|_{x=l} = 0, \qquad t > 0. \tag{2.5}$$

The problem $(2.2),(2.3),(2.4)$ is known as the Dirichlet problem for the diffusion equation, while $(2.2),(2.3),(2.5)$ as the Neumann problem. At first we discuss a property of the diffusion equation , known as the maximum-minimum principle.
let

$R = \{(x,t) : 0 \le x \le l, 0 \le t \le T\}$ be a closed rectangle and

$\mathrm{E} = \{(x,t) \in R : t = 0 \, or \, x = 0 \, or \, x = l\}$.

3. Procedure of solving the problem

In this section, we study the following theorem

Theorem 3.1.

Let $u(x,t)$ be a continuous function in R which satisfies equation (2.2) in $R \setminus \mathrm{E}$. Then

$$\max_R u(x,t) = \max_{\mathrm{E}} u(x,t), \tag{3.6}$$

$$\min_R u(x,t) = \min_{\mathrm{E}} u(x,t), \tag{3.7}$$

$$\max_R a(x,t) = \max_{\mathrm{E}} a(x,t) \tag{3.8}$$

$$\min_R a(x,t) = \min_{\mathrm{E}} a(x,t). \tag{3.9}$$

Proof:

We use the method of contradiction. Assume that the maximum value of $u(x,t)$ attained at an interior point (x_0,t_0). Let $M = \max_{\mathrm{E}} u$, thus there exists a finite $\varepsilon > 0$ such that

$$u(x_0,t_0) = M + \varepsilon$$

Furthermore, at the maximum point (x_0,t_0), we have

$$\frac{\partial u(x_0,t_0)}{\partial x} = 0, \quad \frac{\partial^2 u(x_0,t_0)}{\partial x^2} \le 0, \quad \frac{\partial u(x_0,t_0)}{\partial t} \ge 0$$

In order to show contradiction, we need to rule out the possibility of equality.

Consider $w(x,t) = u(x,t) + \delta(t - t_0)$ for a positive constant $\delta > 0$. At the point (x_0,t_0), we have

$$w(x_0,t_0) = M + \varepsilon$$

Since both $t, t_0 \le T$,

$$\delta(t - t_0) \le \delta T$$

Now we choose δ such that, $\delta T \le \dfrac{\varepsilon}{2}$, Since, $\max_{\mathrm{E}} u = M$, we have

$$\max_{\mathrm{E}} w \le M + \frac{\varepsilon}{2},$$

Since u is continuous, so is w. Thus, w must have a maximum value at some point (x_1,t_1) in the interior

$$(0 < t_1 \le T, 0 < x < l)$$

$$w(x_1,t_1) \ge w(x_0,t_0) = M + \varepsilon$$

Therefore,

$$\frac{\partial^2 w(x_1,t_1)}{\partial x^2} \le 0, \quad \frac{\partial w(x_1,t_1)}{\partial t} \ge 0$$

Since

$$\frac{\partial^2 u(x_1,t_1)}{\partial x^2} = \frac{\partial^2 w(x_1,t_1)}{\partial x^2} \quad \frac{\partial u(x_1,t_1)}{\partial t} = \frac{\partial w(x_1,t_1)}{\partial t} + \delta.$$

We conclude that

$$\frac{\partial^2 u(x_1,t_1)}{\partial x^2} \le 0, \qquad \frac{\partial u(x_1,t_1)}{\partial t} \ge \delta > 0,$$

which is contradictory to

$$\frac{\partial u(x,t)}{\partial t} = a(x,t)\frac{\partial^2 u(x,t)}{\partial x}.$$

Therefore $\max_R u(x,t) = \max_E u(x,t)$ .The same way above we get (3.8).

Considering the function $v(x,t) = -u(x,t)$ we have (3.7), and considering the function $s(x,t) = -a(x,t)$ we get (3.9).

4. Non-homogeneous for the second-order partial differential equation

By maximum-minimum principle it follows the uniqueness of the solution of the Non-homogeneous for the second-order partial differential equation

$$\begin{aligned} &\frac{\partial u(x,t)}{\partial t} - a(x,t)\frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t) && 0 < x < l, 0 < t \le T, \\ &u(x,0) = f(x) && 0 \le x \le l, \\ &u(0,t) = g_1(t), u(l,t) = g_2(t) && 0 \le t \le T. \end{aligned} \tag{4.1}$$

Suppose

$$\begin{aligned} &f(x,t) \in C(R) && f(x) \in C[0,l], \\ &g_1(t) \in C[0,T] && g_2(t) \in C[0,T], \\ &f(0) = g_1(0) && f(l) = g_2(0). \end{aligned} \tag{4.2}$$

By a solution we mean a function $u \in C(R)$ which is differentiable inside R and satisfies the equation along with the initial and the boundary conditions of (4.1).

Theorem 4.1. the problem in (4.1) and (4.2) has no more than one solution.

Proof: suppose $u(x,t)$ and $v(x,t)$ are two solutions of (4.1).

Let $k(x,t) = u(x,t) - v(x,t)$

Then

$$\begin{aligned} &\frac{\partial k(x,t)}{\partial t} - a(x,t)\frac{\partial^2 k(x,t)}{\partial x^2} = 0, && 0 < x < l, 0 < t \le T, \\ &k(x,0) = 0, && 0 \le x \le l, \\ &k(0,t) = k(l,t) = 0, && 0 \le t \le T. \end{aligned}$$

By theorem 3.1 it follows

$$\max_R k(x,t) = \min_R k(x,t) = 0$$
$$\max_R a(x,t) = \min_R a(x,t) = 0$$

Therefore $k(x,t) \equiv 0$, so that

$u(x,t) \equiv v(x,t)$ for every $(x,t) \in R$. ■

Consider the problem (4.2), with $f = g_1 = g_2 = 0$, that is

$$\begin{aligned}
&\frac{\partial u(x,t)}{\partial t} - a(x,t)\frac{\partial^2 u(x,t)}{\partial x^2} = 0 && 0 < x < l, 0 < t \le T,\\
&u(x,0) = \varphi(x) && 0 \le x \le l,\\
&u(0,t) = u(l,t) = 0 && 0 \le t \le T.
\end{aligned} \tag{4.3}$$

As a corollary of theorem 3.1 the continuous dependence of solution of (4.3) with respect to initial data follows.

Corollary 4.1. let $u_i(x,t)$ be a solution of (4.3) with initial data $f_i(x), i = 1,2$. Then

$$\max_{0 \le x \le l} \left| u_1(x,t) - u_2(x,t) \right| \le \max_{0 \le x \le l} \left| f_1(x) - f_2(x) \right| \tag{4.4}$$

For every $t \in [0,T]$

Proof: consider the function $v(x,t) = u_1(x,t) - u_2(x,t)$, which satisfies

$$\begin{aligned}
&\frac{\partial v(x,t)}{\partial t} - a(x,t)\frac{\partial^2 v(x,t)}{\partial x^2} = 0 && 0 < x < l, 0 < t \le T,\\
&v(x,0) = f_1(x) - f_2(x) && 0 \le x \le l,\\
&v(0,t) = v(l,t) = 0 && 0 \le t \le T.
\end{aligned}$$

By theorem 3.1 it follows that

$$\begin{aligned}
u_1(x,t) - u_2(x,t) &\le \max\{\max_{0 \le x \le l}(f_1(x) - f_2(x)), 0\}\\
&\le \max_{0 \le x \le l} \left| f_1(x) - f_2(x) \right|
\end{aligned}$$

And

$$\begin{aligned}
u_1(x,t) - u_2(x,t) &\ge \min\{\min_{0 \le x \le l}(f_1(x) - f_2(x)), 0\}\\
&\ge -\max\{\max_{0 \le x \le l}(f_1(x) - f_2(x)), 0\}\\
&\ge -\max_{0 \le x \le l} \left| f_1(x) - f_2(x) \right|
\end{aligned}$$

Which imply (4.4).■

The uniqueness and stability of solution to (4.3) can be derived by another approach, known as the energy method. Let u be a solution of the problem (4.3). The quantity $H(t) = \int_0^l u^2(x,t)dx$ is referred to as the thermal energy at the instant t . we shall show that H(t) is a decreasing function .

Theorem 4.2.

(a) let $u(x,t)$ be a solution of (4.3).then

$H(t_1) \geq H(t_2),$ if $0 \leq t_1 \leq t_2 \leq T$.

(b) let $u_i(x,t)$ be a solution of (4.3) corresponding to the initial data $f_i(x), i = 1,2$ .then

$$\int_0^l (u_1(x,t) - u_2(x,t))^2 dx \leq \int_0^l (f_1(x) - f_2(x))^2 dx .$$

Proof: (a) Multiplying the equation by u, using

$$u\frac{\partial u(x,t)}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(u^2), \qquad u\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial}{\partial x}(u\frac{\partial u(x,t)}{\partial x}) - \frac{\partial u^2(x,t)}{\partial x}$$

And integrating, we obtain

$$0 = \int_0^l \left(\frac{\partial u(x,t)}{\partial t} - a(x,t)\frac{\partial^2 u(x,t)}{\partial x^2} \right) u dx$$

$$= \int_0^l \left[\frac{1}{2}\frac{\partial}{\partial t}(u^2) - a(x,t)\frac{\partial}{\partial x}(u\frac{\partial u(x,t)}{\partial x}) + a(x,t)\frac{\partial u^2(x,t)}{\partial x} \right] dx$$

$$0 = \frac{1}{2}\frac{d}{dt}\int_0^l u^2(x,t)dx - a(l,t)u(l,t)\frac{\partial u(x,t)}{\partial x}\bigg|_{x=l} + a(0,t)u(0,t)\frac{\partial u(x,t)}{\partial x}\bigg|_{x=0} + \int_0^l a(x,t)\frac{\partial u^2(x,t)}{\partial x} dx$$

Where the last equality is a consequence of the boundary condition (2)

$$0 = \frac{1}{2}\frac{d}{dt}\int_0^l u^2(x,t)dx + \int_0^l a(x,t)\frac{\partial u^2(x,t)}{\partial x} dx$$

$$\frac{dH}{dt}(t) = -2\int_0^l a(x,t)\frac{\partial u^2(x,t)}{\partial x} dx$$

This implies that $\frac{dH}{dt}(t) \leq 0$

Thus H(t) is a non-increasing function of time t, i.e.,

$H(t_1) \geq H(t_2)$ for all $t_2 \geq t_1 \geq 0$.

(b) the function $v(x,t) = u_1(x,t) - u_2(x,t)$ satisfies (4.3) with

$\varphi(x) = \varphi_1(x) - \varphi_2(x)$. Therefore for $t \geq 0$ by (a)

$$\int_0^l (u_1(x,t) - u_2(x,t))^2 \leq \int_0^l (u_1(x,0) - u_2(x,0))^2 dx$$

$$= \int_0^l \left(\varphi_1(x) - \varphi_2(x)\right)^2 dx.$$

Now to show that $\dfrac{d^2 H(t)}{dt^2} = 4\int_0^l u_t^2 dx$

We can multiply by $\dfrac{\partial u(x,t)}{\partial t}$ and integrate with respect to and get

$$\int_0^l \frac{\partial u^2(x,t)}{\partial t} dx = \int_0^l a(x,t) \frac{\partial u(x,t)}{\partial t} \frac{\partial^2 u(x,t)}{\partial x^2} dx$$

$$\int_0^l \frac{\partial u^2(x,t)}{\partial t} dx = \int_0^l a(x,t) \left[\frac{\partial}{\partial x}\left(\frac{\partial u(x,t)}{\partial x} \frac{\partial u(x,t)}{\partial t} \right) - \frac{\partial u(x,t)}{\partial x} \frac{\partial^2 u(x,t)}{\partial x \partial t} \right] dx$$

$$\int_0^l \frac{\partial u^2(x,t)}{\partial t} dx = \int_0^l a(x,t) \frac{\partial}{\partial x}\left(\frac{\partial u(x,t)}{\partial x} \frac{\partial u(x,t)}{\partial t} \right) dx - \int_0^l a(x,t) \frac{\partial u(x,t)}{\partial x} \frac{\partial^2 u(x,t)}{\partial x \partial t} dx$$

$$\int_0^l \frac{\partial u^2(x,t)}{\partial t} dx = a(l,t) \left.\frac{\partial u(x,t)}{\partial x}\right|_{x=l} \left.\frac{\partial u(x,t)}{\partial t}\right|_{x=l} - a(0,t) \left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} \left.\frac{\partial u(x,t)}{\partial t}\right|_{x=0} - \int_0^l a(x,t) \frac{\partial u(x,t)}{\partial x} \frac{\partial^2 u(x,t)}{\partial x \partial t} dx$$

By the chain rule, we get

$$\frac{\partial}{\partial t} \frac{\partial u^2(x,t)}{\partial x} = 2 \frac{\partial u(x,t)}{\partial x} \frac{\partial^2 u(x,t)}{\partial x \partial t}$$

$$\int_0^l \frac{\partial u^2(x,t)}{\partial t} dx = a(l,t) \frac{\partial u(l,t)}{\partial x} \frac{\partial u(l,t)}{\partial t} - a(0,t) \frac{\partial u(0,t)}{\partial x} \frac{\partial u(0,t)}{\partial t} - \frac{1}{2} \int_0^l a(x,t) \frac{\partial}{\partial t} \frac{\partial u^2(x,t)}{\partial x} dx$$

$$\frac{1}{2} \frac{d}{dt} \int_0^l a(x,t) \frac{\partial u^2(x,t)}{\partial x} dx = -\int_0^l \frac{\partial u^2(x,t)}{\partial t} dx + a(l,t) \frac{\partial u(l,t)}{\partial x} \frac{\partial u(l,t)}{\partial t} - a(0,t) \frac{\partial u(0,t)}{\partial x} \frac{\partial u(0,t)}{\partial t}$$

According to the boundary condition (2),

$u(0,t) = u(l,t) = 0$ for all $t > 0$

Since $u(0,t)$ and $u(l,t)$ are constant with respect to time, we conclude that

$u_t(0,t)=u_t(l,t)=0$ for $t>0$.Thus, we get that

$$\frac{d}{dt}\int_0^l a(x,t)\frac{\partial u^2(x,t)}{\partial x}dx=-2\int_0^l \frac{\partial u^2(x,t)}{\partial t}dx$$

$$\frac{d}{dt}\int_0^l -2a(x,t)\frac{\partial u^2(x,t)}{\partial x}dx=4\int_0^l \frac{\partial u^2(x,t)}{\partial t}dx$$

$$\frac{d^2H(t)}{dt^2}=4\int_0^l \frac{\partial u^2(x,t)}{\partial t}dx$$

5. Study Some Applications For Equation (2.1)

5.1. We can solve the problem (2.1) , when $a(x,t)=xt$,with $f(x)=3x$,

$$\begin{aligned}
&\frac{\partial u(x,t)}{\partial t}-xt\frac{\partial^2 u(x,t)}{\partial x^2}=0 && 0<x<2, 0<t,\\
&u(x,0)=3x && 0\le x\le 2,\\
&u(0,t)=u(2,t)=0 && 0\le t.
\end{aligned}$$

$$u(x,t)=\sum_{n=1}^{4} B_n e^{\frac{-a(x,t)n^2\pi^2 t}{l^2}}\sin[\frac{n\pi x}{l}]$$

$$\begin{aligned}
B_n&=\frac{2}{l}\int_0^l f(x)\sin[\frac{n\pi x}{l}]\\
&=\frac{2}{2}\int_0^2 3x\sin[\frac{n\pi x}{2}]dx\\
&=\frac{-12}{n\pi}(-1)^n
\end{aligned}$$

Then the general solution of (5.1) is

$$u(x,t)=\sum_{n=1}^{4}\frac{-12}{n\pi}(-1)^n e^{\frac{-xn^2\pi^2t^2}{4}}\sin[\frac{n\pi x}{2}]$$

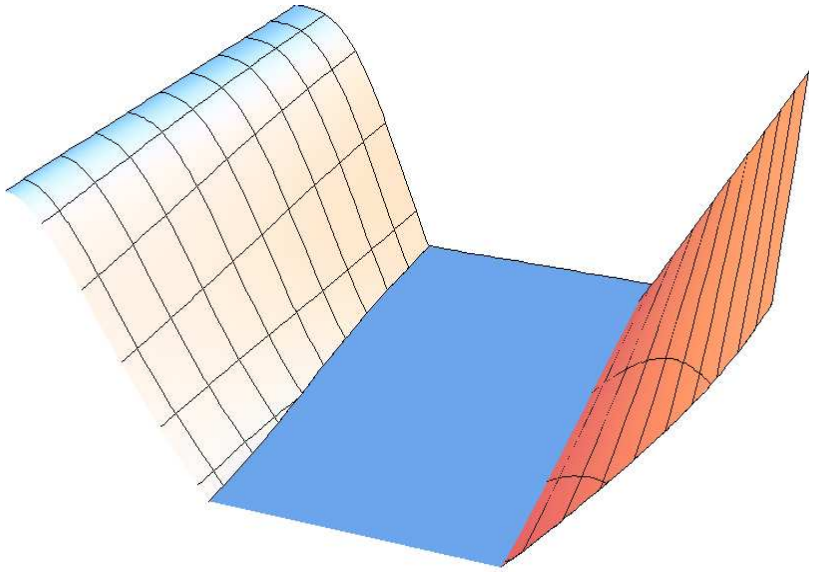

Figure (1) Graph of the function $u = u(x,t)$ in problem (5.1)

5.2. If we consider solve the problem (2.1), when $a(x,t) = \sin(xt)$,with $f(x) = e^{5x}$:

$$\frac{\partial u(x,t)}{\partial t} - \sin(xt)\frac{\partial^2 u(x,t)}{\partial x^2} = 0 \qquad 0 < x < 3, 0 < t,$$
$$u(x,0) = e^{5x} \qquad 0 \le x \le 3,$$
$$u(0,t) = u(3,t) = 0 \qquad 0 \le t.$$

$$u(x,t) = \sum_{n=1}^{4} B_n e^{\frac{-a(x,t)n^2\pi^2 t}{l^2}} \sin[\frac{n\pi x}{l}]$$

$$B_n = \frac{2}{l}\int_0^l f(x)\sin[\frac{n\pi x}{l}]$$

$$= \frac{2}{3}\int_0^3 e^{5x}\sin[\frac{n\pi x}{2}]dx$$

$$= \frac{4n\pi - 4e^{15}n\pi\cos[\frac{3n\pi}{2}] + 40e^{15}\sin[\frac{3n\pi}{2}]}{300 + 3n^2\pi^2}$$

Then the general solution of problem (5.2) is:

$$u(x,t) = \sum_{n=1}^{4} \frac{4n\pi - 4e^{15}n\pi\cos[\frac{3n\pi}{2}] + 40e^{15}\sin[\frac{3n\pi}{2}]}{300 + 3n^2\pi^2} e^{\frac{-\sin[xt]n^2\pi^2 t}{4}} \sin[\frac{n\pi x}{2}]$$

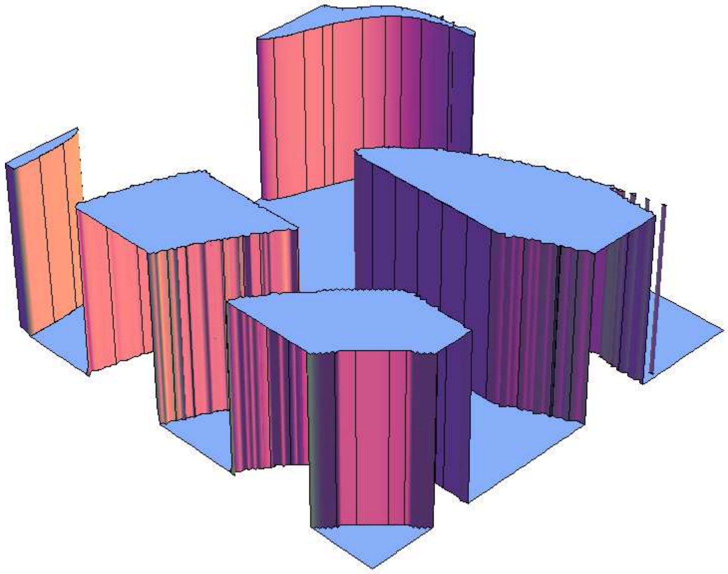

Figure (2) Graph of the function $u = u(x,t)$ in problem (5.2)

5.3. We can solve the problem for the equation (2.1), when $a(x,t) = \sqrt{xt}$,with $f(x) = \sinh(x)$

$$\frac{\partial u(x,t)}{\partial t} - \sqrt{xt}\,\frac{\partial^2 u(x,t)}{\partial x^2} = 0 \qquad 0 < x < 5, 0 < t,$$

$$u(x,0) = \sinh(x) \qquad 0 \le x \le 5,$$

$$u(0,t) = u(5,t) = 0 \qquad 0 \le t.$$

$$u(x,t) = \sum_{n=1}^{4} B_n e^{\frac{-a(x,t) n^2 \pi^2 t}{l^2}} \sin[\frac{n\pi x}{l}]$$

$$B_n = \frac{2}{l}\int_0^l f(x)\sin[\frac{n\pi x}{l}]$$

$$= \frac{2}{5}\int_0^5 \sinh[x]\,\sin[\frac{n\pi x}{5}]dx$$

$$= -\frac{2(-5\cosh[5]\sin[n\pi] + n\pi\cos[n\pi]\sin[5]}{25 + n^2\pi^2}$$

Then the general solution of problem (5.3) is

$$u(x,t) = \sum_{n=1}^{4} -\frac{2(-5\cosh[5]\,\sin[n\pi] + n\pi\cos[n\pi]\,\sin[5]}{25 + n^2\pi^2} e^{\frac{-\sqrt{xt}\, n^2\pi^2 t}{25}} \sin[\frac{n\pi x}{5}]$$

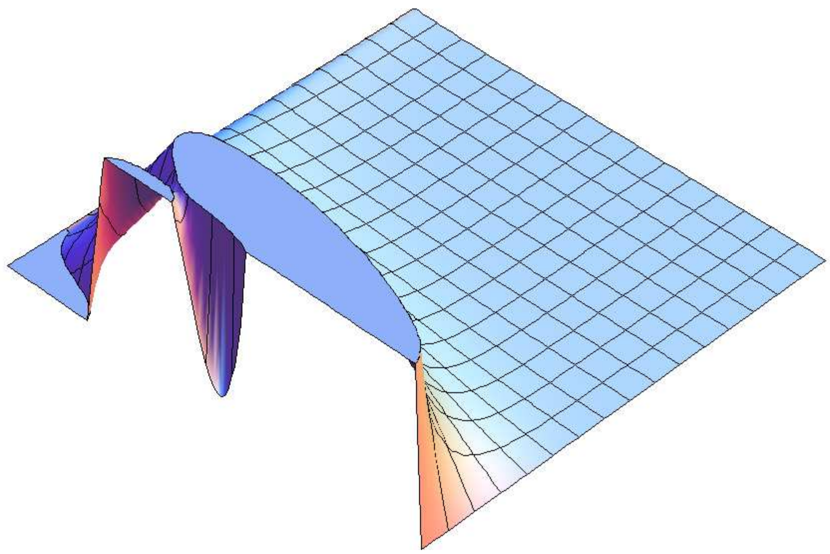

Figure (3) Graph of the function $u = u(x,t)$ in problem (5.3)

Conclusion:

We have developed a heat equation and by relying on a function $a(x,t)$ instead of using constant which is common in all previous studies have reached to the existence and oneness of the solution to the equations (2.1). And then we can apply some examples of scientific importance that confirms the fact our findings.

References:

[1] Khalid Masood , Salim Messaoudi and F.D. Zaman,(2002). Initial inverse problem in heat equation with Bessel operator, International Journal of Heat and Mass Transfer, 45 :2959–2965 .

[2] E. Prulière , F. Chinesta , A. Ammar, A. Leygue , A. Poitou,(2013). On the solution of the heat equation in very thin tapes, International Journal of Thermal Sciences, 65:148–157.

[3] Sining Zheng and Jing Ma,(2009). Non-simultaneous blow-up in coupled heat equations with multi-nonlinearities, Nonlinear Analysis,70:4165-4177.

[4] Yusuke Yamauchi,(2011). Life span of solutions for a semilinear heat equation with initial data having positive limit inferior at infinity, Nonlinear Analysis, 74:5008–5014.

[5] Zhengce Zhang and Bei Hu,(2010). Rate estimates of gradient blowup for a heat equation with exponential nonlinearity, Nonlinear Analysis, 72 :4594_4601.

[6] Bingchen Liu, Fengjie Li,(2009). Optimal classification for blow-up phenomena in heat equations coupled via exponential sources, Nonlinear Analysis , 71:1263_1270.

[7] Daoud S. Daoud,(2008). Determination of the source parameter in a heat equation with a non-local boundary condition, Journal of Computational and Applied Mathematics, 221:261–272.

[8] Hamid R. Ghazizadeh , A. Azimi , M. Maerefat,(2012). An inverse problem to estimate relaxation parameter and order of fractionality in fractional single-phase-lag heat equation, International Journal of Heat and Mass Transfer, 55:2095–2101.

[9] Nakao Hayashi , Elena I. Kaikina , Pavel I. Naumkin,(2007). Subcritical nonlinear heat equation, J. Differential Equations, 238:366–380.

[10] A. B. Vasileva, and V.F. Butuzob, Asymptotic Methods in Theory of singular Perturbations, Moscow, 1990.

[11] V. B. Levenshtam, Construction of higher approximations of the averaging method for parabolic initial-boundary –value problems by the method of boundary layers, Izv. Vyssh. Uchebn. Zaved. Mat., vol. 48.no. 3, pp. 41-45, 2004.

[12] V. B. Levenshtam and H. J. Abood, Asymptotic integration of the problem on heat distribution in a thin rod with rapidly varying sources of heat, Journal of Mathematical sciences, vol. 129, no. 1, pp. 3626-3634, 2005.

[13] A. K. Kapikyan and V. B. Levenshtam, First-order partial differential equations with large high-frequency terms, Computational Mathematics and Mathematics physics, vol. 48,on. 11, pp. 2059-2076,2008.

[14] V. B. Levenshtam, Asymptotic expansions of periodic solutions of ordinary differential equations with large high-frequency terms, Differential Equations, vol 44, no. 1, pp. 54-79, 2008.

[15] V. F. Butuzov and E. A. Derkunova, On a Singularly Perturbed First-Order Partial Differential Equation in the Case of Intersecting Roots of the Degenerate Equation, Differential Equations, vol. 45,No. 2, pp. 186-196.2009.

[16] E. V. Krutenko and V. B. Levneshtam, Asymptotics of a solution to a second order linear differential equation with large summands, Siberian Mathematical Journal, vol. 51 , no. 1, pp. 57-71.2010.

[17] V. F. Butozov and A. V. Kostin, On a Singularity Perturbed System of Two Second-Order Equations in the Case of Intersecting Roots of the Degenerate Equation, Differential Equations, vol. 45,no. 7, pp. 933-950.2009.

[18] H. J. Abood, Asymptotic Integration Problem of Periodic Solutions of Ordinary Differential Equations Containing a Large High-Frequency Terms, Dep. In VINITI, no., 338-B2004, 28 Pages, Moscow, Russian, 26.02.2004.

[19] H. J. Abood, Asymptotic Integration Problem of Periodic Solutions of Ordinary Differential Equation of Third Order with Rapidly Oscillating Terms, Dep. In VINITI, no. 1357-B2004, 32 pages, Moscow, Russian, 05.08.2004.