The Existence Solution to the Development Wave Equation With Arbitrary Conditions

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Abstract

We study the development wave equationwith some conditions and proving the existence and uniqueness solutionbyusing the reflection method .

Keywords: Wave equation, Mathematica, reflection method**.**

1. Introduction

The first-order asymptotic form is obtained and proved for the solution of a system of two partial differential equations with small parameters in the derivatives for the regular part, two boundary-layer parts and corner boundary part. Some results on the construction of asymptotics for solutions of singularly perturbed problem with fist-order partial derivatives can be found in [1] and [2]. For a singularly perturbed system of two second-order differential equations (one rapid and one slow), they are proved the existence of a solution and obtain its asymptotics for the case in which the degenerate equation has two intersecting roots in [3]and[4]. In[5], proposed an algorithm of asymptotic integration of the initial-boundary-value problem for the heat-conduction equation with minor terms (nonlinear sources of heat) in a thin rod of thickness $\varepsilon = \omega^{-1/2}$ oscillating in time with frequency ω^{-1}.In[6],considered nonlinear systems of singularly perturbed integro-differential equations with fast varying kernels, it is assumed that the spectrum of the limiting operator lies in the closed left half-plane $\mathrm{Re}\,\lambda \leq 0$, and derived an algorithm for obtaining regularized (in the sense of Lomov) asymptotic solutions in both the no resonance and resonance cases. In [7], considered the second order ordinary differential equations whose coefficients of the unknowns contain smooth and rapidly oscillating summands proportional to the positive powers of the oscillation frequency. Examples [8-11], of applications of second order partial differential equations in modeling can be found in elasticity and packed-bed electrode .In[12], to predict the wave propagation in a given region over time , it is often necessary to find the numerical solution for wave equation. With the techniques of discrete differential calculus , they proposed two unconditional stable numerical schemes for simulation wave equation on the space manifold and the time. The integral bifurcation method is used to study a nonlinearly dispersive wave equation of Camasa- Holm type. Loop soliton solution and periodic loop soliton solution, solitary wave solution and solitary cusp wave solution, smooth periodic wave soliton and non-smooth periodic wave solution of this equation are obtained, their dynamic characters are discussed in [13]. They are investigate the initial value problem for a semi-linear wave equation in n-dimensional space based on the decay estimate of solutions to the corresponding linear equation, they defined a set of time-weighted Sobolev space. Under small condition on the initial value, and proved the global existence and asymptotic behavior of the solution in the corresponding Sobolev spaces by the contraction mapping principle in [14]. In [15], they studied a dispersive counterpart of the classical gas dynamics problem of the interaction of a shock wave with a counter-propagating simple rarefaction wave, often referred to as the shock wave refraction. The refraction of a one-dimensional dispersive shock wave due to its head –on collision with the centered rarefaction wave is considered in the framework of the defocusing nonlinear Schrödinger equation . The present a new mesh less method developed by combining the quasi-linear method of fundamental solution and the finite difference method to analyze wave equations. The method of fundamental solution is an efficient numerical method for solution Laplace equation for both two- and three-dimensional problems. The method has also been applied for the solution of Poisson equations and transient Poisson-type equations by finding the particular solution to the non-homogeneous term . In general ,approximate particular solutions are constructed using the interpolation of the non-homogeneous term by the radial basis function in [16]. The element –free Galerkin method is a promising method for solving partial differential equations in which trial and test functions employed in the discretization process result from moving least-squares approximation, by employing the improved moving least-squares approximation they are derived formulae for an improved element-free Galerkin method for the modified equal width wave equation in [17]. Our principal aim in the present paper is concerned the development wave equationwith initial boundary conditions and proving the existence and uniqueness solutionbyusing the reflection method.

2.Statement of the problem

We study the second-order partial differential equation define for the following problem:

$$\frac{\partial^2 u(x,t)}{\partial t^2}=a^2(x,t)\frac{\partial^2 u(x,t)}{\partial x^2}, \quad (2.1)$$

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0}=\psi(x), \qquad x\in R$$
$$\left.u(x,t)\right|_{t=0}=\varphi(x), \qquad x\in R \qquad (2.2)$$

where x signifies the spatial variable or "position" , t the "time" variable ,$u(x,t)$ the unknown continuous function and $a(x,t)$ is an arbitrary continuous function dependent with respect variable x and t. Physically $u(x,t)$ represents the "value" of the normal displacement of a particle at position x and time t .The initial boundary conditions are hold in equations (2.2), where The functions $\psi(t)$ and $\varphi(x)$ are continuous and infinitely differentiable with respect to each of their arguments.In what follows we construct and justify of initial boundary-value type for the solution of the problem (2.1) and (2.2) , subject to the following requirements:

I. The function $a^2(x,t)>0$ for all $x\in R$, $t>0$

II. $\varphi_1(0)=\psi_1(0), a(0,0)^2=\psi'(0)$, ensures the compatibility of the initial and boundary functions at the corner point $(0,0)$ and $(1,0)$.

3. Procedure of solving the problem

The characteristic equation of (2.1) is

$$(dx)^2 = a(x,t)(dt)^2$$

and

$$\left.\begin{cases} x + a(x,t)\, t = a_1 \\ x - a(x,t)\, t = a_2 \end{cases}\right\}.$$

Are two families of real characteristics . . Introducing the new variables

$$\Phi : \begin{cases} \xi = x + a(x,t)t \\ \eta = x - a(x,t)t \end{cases}, \qquad \Phi^{-1} : \begin{cases} x = (\xi+\eta)/2 \\ t = (\xi-\eta)/2a(x,t) \end{cases}$$

We define the following functions:

$$U(x + a(x,t), x - a(x,t)t,$$

$$U(\xi,\eta) = u(((\xi+\eta)/2, (\xi-\eta)/2a(x,t)).$$

The equation (2.1) reduces to

$$U_{\xi\eta}(\xi,\eta) = 0. \qquad (2.3)$$

Therefore

$$U_{\xi\eta}(\xi,\eta) = F(\xi),$$

$$U(\xi,\eta) = \int F(\xi)\, d\xi + g(\eta) = f(\xi) + g(\eta).$$

And in the original variables $u = u(x,t)$ is of the form

$$u(x,t) = f(x + a(x,t)t) + g(x - a(x,t)t). (2.4)$$

Known as the general solution of (2.1) .It is the sum of the function $g(x - a(x,t)t)$ which presents a shape traveling without change to the right with speed c and the function $f(x + a(x,t)t)$. Another shape , traveling to the left with speed $a(x,t)$.Consider the Cauchy (initial value) problem for (2.1)

$$\frac{\partial^2 u}{\partial t^2} = a^2(x,t)\frac{\partial^2 u}{\partial x^2}, \qquad x \in R, t > 0$$

$$u(x,0) = \varphi(x), \quad x \in R \qquad (2.5)$$

$$\frac{\partial u}{\partial t}(x,0) = \psi(x), \quad x \in R$$

where φ and ψ are continuous arbitrary functions with respect to x . Further we denote $R^+ = \{t : t \geq 0\}$. Then the following theorem is holds:

Theorem(1): If $\varphi \in c^2(R)$ and $\psi \in c^1(R)$ the problem (2.5) has a unique solution $u \in c^2(R \times R^+)$ given by the formula:

$$u(x,t) = \frac{1}{2}\left(\varphi\left(x + a(x,t)t\right) + \varphi\left(x - a(x,t)t\right)\right) + \frac{1}{2a(x,t)}\int_{x-a(x,t)t}^{x+a(x,t)t}\psi(s)ds \quad (2.6)$$

Proof: We are looking for a solution of the problem in the form (1.3) satisfying the initial conditions at $t = 0$

$$f(x) + g(x) = \varphi(x), \quad (2.7)$$

$$a(x,t)f'(x) - a(x,t)g'(x) = \psi(x). \quad (2.8)$$

Differentiating (2.7) with respect to x and solving the linear system for f' and g', we obtain

$$f'(x) = \frac{1}{2}\varphi'(x) + \frac{1}{2a(x,t)}\psi(x) \quad (2.9)$$

$$g'(x) = \frac{1}{2}\varphi'(x) - \frac{1}{2a(x,t)}\psi(x) \quad (2.10)$$

Integrating (2.9) and (2.10) from 0 to x we get

$$f(x) = \frac{1}{2}\varphi(x) + \frac{1}{2a(x,t)}\int_0^x \psi(s)ds + \left(f(0) - \frac{1}{2}\varphi(0)\right)$$

$$g(x) = \frac{1}{2}\varphi(x) + \frac{1}{2a(x,t)}\int_0^x \psi(s)ds + \left(g(0) - \frac{1}{2}\varphi(0)\right)$$

Using equation (2.7), we have $f(0) + g(0) = \varphi(0)$. Therefore;

$$u(x,t) = f\left(x + a(x,t)t\right) + g\left(x - a(x,t)t\right)$$

$$= \frac{1}{2}\left(\varphi\left(x + a(x,t)t\right) + g\left(x - a(x,t)t\right)\right) + \frac{1}{2a(x,t)}\int_0^{x+a(x,t)t}\psi(s)ds - \int_0^{x-a(x,t)t}\psi(s)ds$$

$$= \frac{1}{2}\left(\varphi\left(x + a(x,t)t\right) + \varphi\left(x - a(x,t)t\right)\right) + \frac{1}{2a(x,t)}\int_{x-a(x,t)t}^{x+a(x,t)t}\psi(s)ds.$$

Conversely, it is easy to see that for $\varphi \in c^2(R)$ and $\psi \in c^1(R)$ this formula gives the solution $u \in c^2(R \times R^+)$ of problem (2.5) . Note that if $\varphi = \psi = 0$, then it follows $u = 0$. The proof of the theorem is complete.

We are study some corollaries from theorem (1) are as follows:

I. Domain of dependence . The value of u at (x_0, t_0) is determined by the restriction of initial functions φ and ψ in the interval $[x_0 - a(x,t)t_0, x_0 + a(x,t)t_0]$ on the x-axis, whose end-points are cut out by the characteristics :

II. $x - x_0 = \pm a(x,t)(t - t_0)$, through the point (x_0, t_0).

The characteristic triangle $\mathrm{V}(x_0, t_0)$ is defined as the triangle in $\boldsymbol{R \times R^+}$ with vertices

$$A_0(x_0 - a(x,t)t_0, 0),\ B_0(x_0 + a(x,t)t_0, 0), P_0(x_0, t_0),\ \forall (x_1, t_1) \in \psi(x_0, t_0).$$

We can obtain

$$[x_1 - a(x,t)t_1, x_1 + a(x,t)t_1] \subset [x_0 - a(x,t)t_0, x_0 + a(x,t)t_0],$$

$$\mathrm{V}(x_1, t_1) \subset \psi(x_0, t_0).$$

and $u(x_1, t_1)$ is determined by the values of φ and ψ on $[x_1 - a(x,t)t_1, x_1 + a(x,t)t_1]$,domain of influence .

The point $(x_0, 0)$ on the x-axis influences the value of u at (x,t) in the wedge –shaped region

$$I(x_0) = \{(x,t) : x_0 - a(x,t)t \le x \le x_0 + a(x,t)t, t \ge 0\}.$$

For any

$$P_1(x_1, t_1) \in I(x_0), \mathrm{V}(x_1, t_1) \mathrm{I}\ I(x_0) \ne \phi,$$

$$P_1(x_1, t_1) \notin I(x_0),\ \mathrm{V}(x_1, t_1) \mathrm{I}\ I(x_0) = \phi.$$

III.For $v(x) \in a^2(R)$ and $w(x,t) \in a^2(R \times [0, \infty))$ introduce uniform norms $\|v\|_\infty = \sup_{x \in R} |v(x)|$.

and $\|w\|_{\infty,T} = \sup_{x \in R, 0 \le t \le T} |w(x,t)|$.

For a given T > 0 by (2.6) it follows :

$$\|u\|_{\infty,T} \le \frac{1}{2}\left(\|\varphi\|_\infty + \|\varphi\|_\infty\right) + \frac{1}{2a(x,t)}\|\psi\|_\infty \int_{x - a(x,t)t}^{x + a(x,t)t} ds$$

$\leq \|\varphi\|_\infty + T\ \|\psi\|_\infty$.

Then for any $\varepsilon > 0$ there exists $\|u\|_{\infty,T} < \varepsilon$ and $\|\psi\|_\infty < \delta$ it follows $\|u\|_{\infty,T} < \varepsilon$, which proves the continuous dependence .

4.Study some applications for equations (2.1) and (2.2)

4.1. Let consider the problem (2.1) and (2.2), with $c = e^{xt}, \psi = 0$ and

$$\varphi(x) = \begin{cases} \dfrac{\sin x}{2x} & if\ x \in (-\pi, \pi) \\ 0 & if\ x \notin (-\pi, \pi) \end{cases}$$

Solution: The solution of the problem is $u(x,t) = \dfrac{1}{2}(\varphi(x + e^{xt}t) + \varphi(x - e^{xt}t)).(4.1)$

Using Mathematica the profile of $u(x,t)$ is presented in Figure (1) at successive instants $t = 0, 1, \pi, 3, 4, 5.$ Note that at t=0 the amplitude is 1. After the instant $t = \pi$ the profile breaks up into two traveling waves moving in opposite directions with speed e^{xt} and amplitude $\dfrac{1}{2}$.The surface $u = u(x,t)$ is presented in figure (2) .We use the Mathematica program:

```
u[x_,t_]:=(f[x+©x t t]+f[x-©x t  t])/2
f[x_]:=Which[-π<=x<=π,Sin[x]/2x,True,0]
h0=Plot[Evaluate[u[x,0]],{x,-5,5},
PlotRange->{0,1.5},PlotLabel->"Wave at t=0"]
h1=Plot[Evaluate[u[x,1]],{x,-5,5},
PlotRange->{0,1.5},PlotLabel->"Wave at t=1"]
h2=Plot[Evaluate[u[x,2]],{x,-5,5},PlotRange->{0,1},PlotLabel-
>"Wave at t=π"]
h3=Plot[Evaluate[u[x,3]],{x,-5,5},
PlotRange->{0,1.5},PlotLabel->"Wave at t=3"]
h4=Plot[Evaluate[u[x,4]],{x,-5,5},
PlotRange->{0,1.5},PlotLabel->"Wave at t=4"]
h5=Plot[Evaluate[u[x,5]],{x,-5,5},
PlotRange->{0,1.5},PlotLabel->"Wave at t=5"]
Show[GraphicsArray[{{h0,h1},{h1,h3},{h4,h5}}],
Frame→True,FrameTicks→None]
Plot3D[u[x,t],{x,-5,5},{t,0,5},PlotPoints→40,
AxesLabel→
PlotRange→{0,1.5}Shading→False]
```

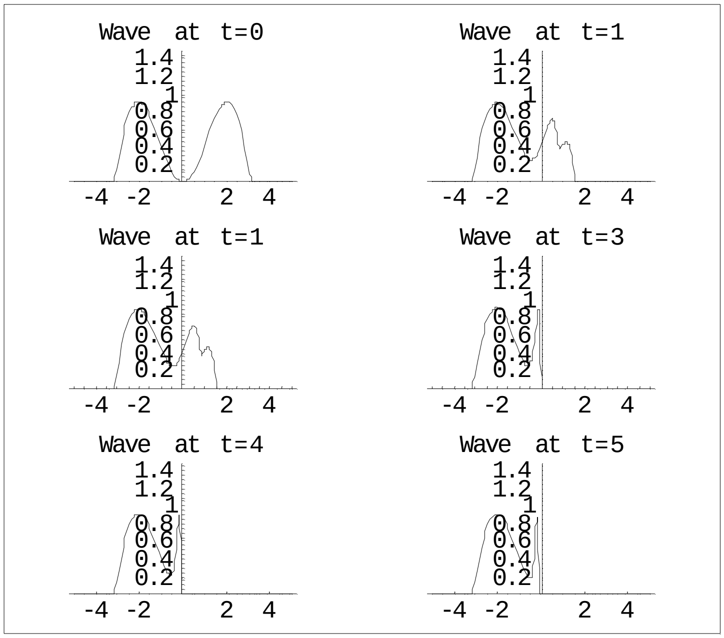

Figure (1) The wave at instant $t = 0, 1, \pi, 3, 4, 5$

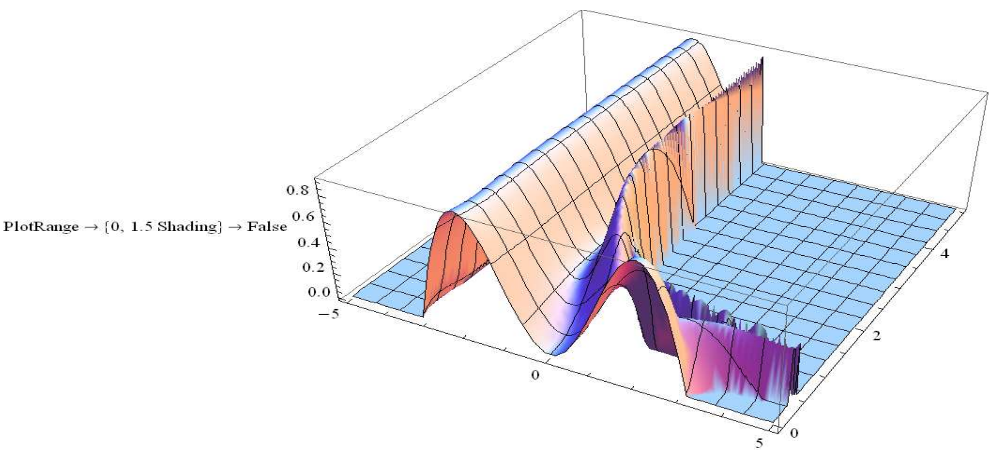

Figure (2)Graph of the function $u = u(x, t)$ in problem (4.1).

4.2. Wecan Solve the problem (2.1) and (2.2), with $c = \frac{1}{xt}, \psi = 0$ and

$$\varphi(x) = \begin{cases} \sin^3 x & \text{if } x \in \left(-\frac{2\pi}{3}, \frac{2\pi}{3}\right) \\ 0 & \text{if } x \notin \left(-\frac{2\pi}{3}, \frac{2\pi}{3}\right) \end{cases}$$

The solution of the problem is:

$u(x,t)=\frac{1}{2}(\varphi(x+(\frac{1}{xt})t)+\varphi(x-(\frac{1}{xt})t))$ (4.2) Using Mathematica the profile of $u(x,t)$ is presented in figure (3) at successive instants $t=0,1,\frac{2\pi}{3},3,4,5.$ Note that at t=0 the amplitude is 1. After the instant $t=\frac{2\pi}{3}$ the profile breaks up into two traveling waves moving in opposite directions with speed $\frac{1}{xt}$ and amplitude $\frac{1}{2}$.The surface $u=u(x,t)$ is presented in figure (4).We use the Mathematica program:

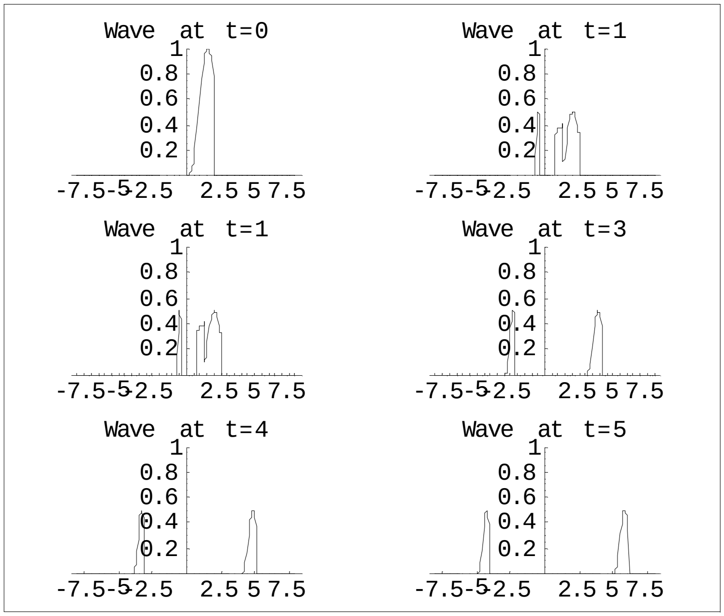

Figure (3) The wave at instant $t=0,1,\frac{2\pi}{3},3,4,5$

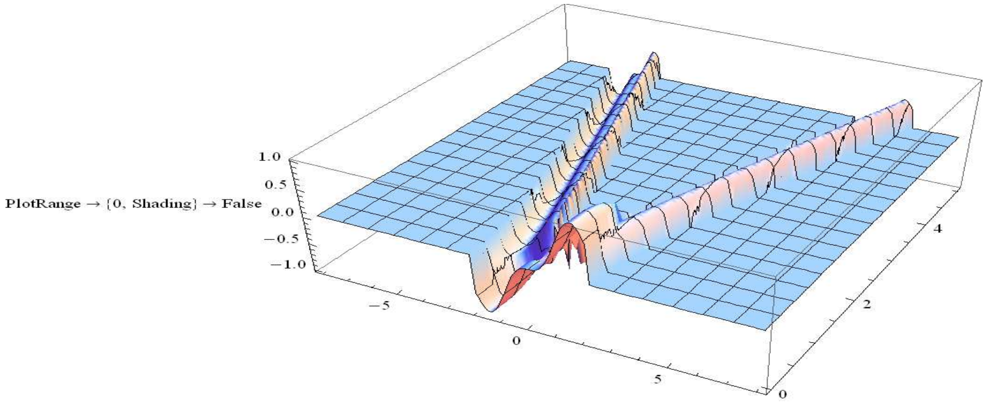

Figure (4)Graph of the function $u=u(x,t)$ in Problem (4.2).

4.3. Now, we study the problem(2.1) and (2.2), with $c=e^{xt}$, $\varphi=0$ and $\psi(x)=\sin(x)^2$.

If $\varphi = 0$, then the solution of $u(x,t)$ can be expressed:

$$u(x,t) = \frac{1}{2e^{xt}} \int_{x+e^{xt}t}^{x+e^{xt}t} \sin(x)^2$$

$$= \frac{1}{2} e^{-xt} (e^{xt} t - \frac{1}{2} \cos(2x) \sin(2e^{xt}) t) \quad (4.3)$$

Using Mathematica the profile of $u(x,t)$ is presented in figure (5) at the successive instants $t = 0,1,2,3.$ note that at $t = 1$ the amplitude is $1/2$ and it remains the same for all next instants . The surface $u = u(x,t)$ is plotted in figure (6) . We can use the following program:

```
 u[x_,t_]:=1/2 ©-t x (©t x t-1/2 Cos[2 x] Sin[2 ©t x t])
h0=Plot[Evaluate[u[x,0]],{x,-8,8},
PlotRange->{0,0.5},PlotLabel->"Wave at t=0"]
h1=Plot[Evaluate[u[x,1]],{x,-8,8},
PlotRange->{0,0.5},PlotLabel->"Wave at t=1"]
h2=Plot[Evaluate[u[x,2]],{x,-8,8},
PlotRange->{0,0.5},PlotLabel->"Wave at t=2"]
h3=Plot[Evaluate[u[x,3]],{x,-8,8},
PlotRange->{0,0.5},PlotLabel->"Wave at t=3"]
Show[GraphicsArray[{{h0,h1},{h2,h3}}],
Frame→True,FrameTicks→None]
Plot3D[u[x,t],{x,-8,8},{t,0,3},
 AxesLabel→PlotPoints→20
PlotRange→{0,0.5}Shading→False]
```

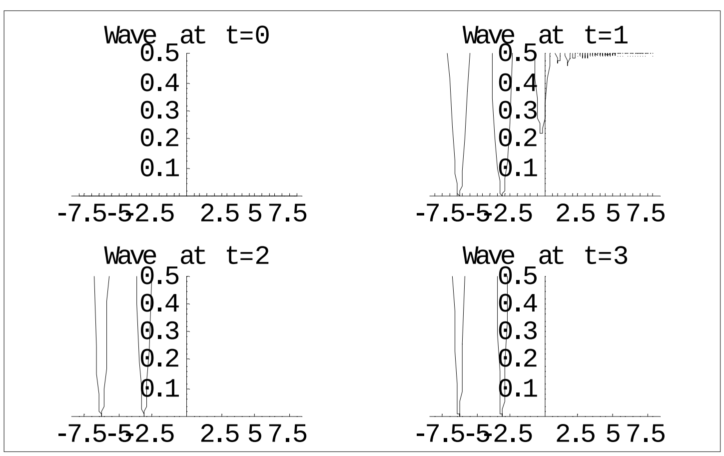

Figure (5)Wave at instants t=0,1,2,3.

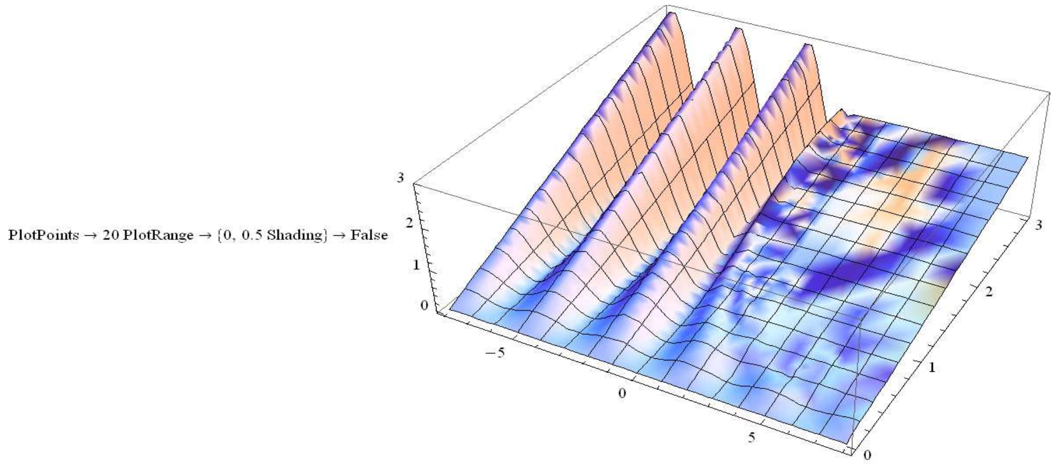

Figure (6)Graph of the function $\frac{1}{2}e^{-xt}(e^{xt}t-\frac{1}{2}\cos(2x)\sin 2\left(e^{xt}\right)t)$

Conclusion

Through the use of the relationship (1.2) with (2.2) the search on the board and using the equation (2.6) proved the uniqueness of the solution as well as possible give example applied different confirm that we have reached so users program.

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