Principally Dual Stable Modules

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Abstract.

Another generalization of fully d-stable modules, in this paper was introduced. A module is principally d-stable if every cyclic submodule of it is d-stable. Quasi-projective principally d-stable module is fully d-stable. For finitely generated modules over Dedekind domains the two concepts (full and principal) d-stability of modules coincide. For regular modules over commutative rings, principal d-stability of modules is equivalent to commutativity and full d-stability of there endomorphism rings.

Keywords: fully (principally) d-stable module; quasi-projective, duo, regular module; Dedekind domain; endomorphism ring; hollow module; exchange property.

1. Introduction.

In two previous papers ([2] and [3]), we introduced the concept of fully d-stable modules and studied some generalizations of it. A submodule $N$ of an $R$-module $M$ is said to be d-stable if $N \subseteq \text{Ker}(\alpha)$ for every homomorphism $\alpha : M \rightarrow M/N$, the module $M$ is said to be fully d-stable, if each of its submodules is d-stable [2]. Full d-stability is dual to the concept of full stability introduced by Abbas in [1], and both of these concepts are stronger than duo property of modules. A submodule $N$ of an $R$-module $M$ is said to be stable if $f(N) \subseteq N$, for any homomorphism $f : N \rightarrow M$, a module is fully stable of all of its submodules are stable [1]. In [1], it was proved that a module is fully stable if and only if each cyclic submodule is stable. Unfortunately it is not the case in full d-stability. This motivates introducing the concept of principally d-stable module which is a generalization of full d-stability. A module will be called principally d-stable if every cyclic submodule of it is d-stable. In this paper we studied this new concept and the conditions that make a principally d-stable module into a fully d-stable. In section 2 main properties of principal d-stable were investigated in addition, we see that quasi-projectivity is a sufficient condition for a principal d-stable module to be fully d-stable. Also we show that over Dedekind domain and integral domain with certain conditions, the two concepts, full (and principal) d-stability coincide. Links between the two dual concepts full stability and full d-stability, in certain conditions, also, was found . In section 3, under regular modules (in some sense), many characterizations to principally d-stable module, via endomorphism rings, were investigated.

Throughout, rings are associative having an identity( unless we state) and all modules are unital. $R$ is a ring and $M$ is a left $R$-module (simply we say module).

2. principally d-stable modules

Definition 2.1. A module is said to be principally d-stable if each of its cyclic submodule is d-stable.

Proposition 2.2. Any quasi-projective principally d-stable is fully d-stable.

Proof: By (Proposition 3.6. [2]).
Proposition 2.3. Every principally d-stable module is duo.

Proof: Let \( M \) be an \( R \)-module, \( f \) an endomorphism of \( M \), and \( N \) a submodule of \( M \). Let \( x \in N \), \( \pi \), be the natural epimorphism of \( M \) onto \( M/Rx \) and \( \alpha = \pi_x \circ f \), then by assumption \( \alpha(x) = 0 \), hence \( f(x) \in Rx \subset N \), that is \( f(N) \subset N \).

Definition 2.4. A ring \( R \) is right (left) principally d-stable if \( R_R ( R ) \) is principally d-stable.

The rings in this paper are assumed to have identity, this makes the concepts of duo, fully d-stability and principal d-stability coincide for rings. Note that a ring is right (left) duo if and only if every right (left) ideal is two sided ideal.

Proposition 2.5. A ring \( R \) is right (left) principally d-stable if and only if it is right (left) fully d-stable.

Proof: The (if part) is clear. We will prove the (only if part, the left case).

Assume that \( R \) is left principally d-stable, \( I \) a left ideal of \( R \) and \( \alpha : R \rightarrow R/I \) is an \( R \)-homomorphism. By assumption and the note before the proposition, \( I \) is a two sided ideal too, if \( x \in I \) then \( \alpha(x) = x\alpha(1) = xPA_1 \), since \( xPA_1 \in I \). Therefore, \( R \) is left fully d-stable.

In [3] we introduced minimal d-stable modules in which minimal submodules are d-stable. Since any minimal submodule is cyclic, so we conclude that any Principally d-stable module is minimal d-stable. The converse of this result is not true, as the \( Z \)-module \( I \) is minimal d-stable (trivially) but not principally d-stable (see remark 2.14).

Another condition which versus principal d-stability into full d-stability is in the following. First we need to introduce the following concept.

Definition 2.6. An \( R \)-module, \( M \) is said to have the quotient embedding property (qe-property, for short) if \( M/N \) can be embedded into \( M/Rx \) for each submodule \( N \) of \( M \) and each \( 0 \neq x \in N \).

Remark 2.7. Let \( M \) be an \( R \)-module. If \( M/Rx \) is semisimple for each \( 0 \neq x \in M \), then \( M \) has the qe-property. In particular every semisimple module has the qe-property.

Proof: If \( x \in N \), where \( N \) is a submodule of \( M \), then there is a natural epimorphism \( \delta : M/Rx \rightarrow M/N \) \((a + Rx \leftrightarrow a + N)\) with \( ker(\delta) = N/Rx \). Since \( M/Rx \) is semisimple, \( N/Rx \) is a direct summand of \( M/Rx \), that is, \( \delta \) is split epimorphism, hence \( \delta \) has a right inverse which is a monomorphism from \( M/N \) into \( M/Rx \).

Proposition 2.8. Let \( M \) be a principally d-stable \( R \)-module. If \( M \) has the qe-property, then \( M \) is fully d-stable.

Proof: Assume that \( M \) is a principally d-stable module, \( \alpha : M \rightarrow M/N \) is an \( R \)-homomorphism, where \( N \) is a submodule of \( M \). Let \( x \in N \), then by hypothesis there is a monomorphism \( \beta : M/N \rightarrow M/Rx \). Now \( \beta \alpha \) is an \( R \)-homomorphism from \( M \) into \( M/Rx \), so \( Rx \subset ker(\beta \alpha) = ker(\alpha) \), since \( \beta \) is a monomorphism. Hence \( N \subset ker(\alpha) \), since \( x \) is an arbitrary element of \( N \), and then \( M \) is fully d-stable.

From Proposition 2.8 and Remark 2.7 we conclude that, if \( M \) is principally d-stable and \( M/Rx \) is semisimple for each \( x \in M \) (or \( M \) itself is semisimple), then \( M \) is fully d-stable.

Note that the \( Z \)-module \( Z \) has the qe-property, but \( Z/4Z \) (for example) is not semisimple. On the other hand \( Z_{(p^\infty)} \) has the qe-property, which is not principally d-stable (see Remark 2.14). So we restate Proposition 2.8 in this way.
Corollary 2.9. Let \( \mathcal{M} \) be a module, with the qe- property. Then the following two statements are equivalent:

(i) \( \mathcal{M} \) is principally d-stable.

(ii) \( \mathcal{M} \) is fully d-stable.

Note that the \( \mathbb{Z} \)-module \( \mathcal{M} = \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}) \) does not satisfy qe-property, since if \( x = (0,1) \), \( N = 2\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}) \), then \( \mathcal{M} = 0 \oplus (\mathbb{Z}/2\mathbb{Z}) \) and \( \mathcal{M}/N \) cannot be embedded in \( \mathcal{M}/\mathbb{Z}x \), on the other hand \( \mathcal{M} \) is not principally d-stable (it is not duo), see Lemma 2.18 below.

Other condition can be regarded to deduce full d-stability from principal d-stability.

Theorem 2.10. Let \( \mathcal{M} \) be an \( R \) -module, with the property that any proper submodule of \( \mathcal{M} \) is contained in a cyclic submodule. Then \( \mathcal{M} \) is fully d-stable if it is principally d-stable.

**Proof:** Let \( \mathcal{N} \) be a submodule of \( \mathcal{M} \) contained in \( \mathcal{R}x \) (for some \( x \in \mathcal{M} \)), then \( \mathcal{N} \) is d-stable in \( \mathcal{R}x \) (since \( \mathcal{R}x \) is cyclic module and hence fully d-stable \([2]\)), also \( \mathcal{R}x \) is d-stable in \( \mathcal{M} \) (since \( \mathcal{M} \) is principally d-stable). Then by transitive property of d-stability (see \([2]\)) \( \mathcal{N} \) is d-stable in \( \mathcal{M} \). Therefore \( \mathcal{M} \) is fully d-stable.

Note that the condition of Theorem 2.10 and the qe-property are independent (although they have the same effect on principally d-stable modules). In the next example a module satisfying the qe-property but not the other will be discussed, while in example 2.12 a module having the property of Theorem 2.10 will be given that does not satisfy qe-property.

In \([2]\) we constructed an example of fully d-stable module which not quasi-projective, in the following, with the help of Corollary 2.9, an other example of a module which is not quasi-projective will be shown it is fully d-stable, first we prove it is principally d-stable and then it satisfies the qe-property. The direct proof of full d-stability is certainly more difficult.

**Example 2.11.** Let \( \mathcal{M} = \{a/b \in \mathbb{Q} \mid b \text{ is square free}\} \), the following properties can be observed for \( \mathcal{M} \):

1. \( \mathcal{M} = \sum_{p \in \text{PR}} \mathbb{Z}^{-1} \), where PR is the set of all prime numbers. (clear)

2. \( \mathcal{M} \) is a torsion-free uniform (not finitely generated) \( \mathbb{Z} \)-module. (clear)

3. \( \mathcal{M} \) is duo. \([10]\)

4. \( \mathcal{M} \) is not quasi-projective.

**Proof:** Recall the following fact from \([11]\), "Any torsion-free quasi-projective module over a Dedekind domain, which is not a complete discrete valuation ring, is torsionless" (Lemma 5.2, \([11]\)). We will show that \( \mathcal{M} \) is not torsionless. (Recall that an \( \mathbb{R} \)-module \( \mathcal{M} \) is torsionless if each non-zero element of \( \mathcal{M} \) has non-zero image under some \( \mathbb{R} \)-linear functional \( f \in \text{Hom}_\mathbb{R}(\mathcal{M}, \mathbb{R}) \). \([8]\))

Let \( f : \mathcal{M} \to \mathbb{Z} \) be a \( \mathbb{Z} \)-homomorphism and \( f(1) = n \neq 0 \), let \( q \) be any prime not dividing \( n \), then
\[ n = f\left(\frac{q}{q}\right) = qf\left(\frac{1}{q}\right) = qk \text{ a contradiction. Hence } f(I) = 0 \text{ for each } f \in \text{Hom}_Z(M, Z). \quad \diamondsuit \]

5. \( M \) is principally d-stable.

**Proof:** Any cyclic submodule of \( M \) is of the form \( Z\frac{a}{b} \), \( \frac{a}{b} \in M \). Since a cyclic submodule is fully d-stable module and if it is d-stable in \( M \), then all its submodules are d-stable in \( M \) by transitive property of d-stability (see [2]), also, since \( Z\frac{a}{b} \subseteq Z\frac{1}{b} \), it is enough to prove that \( Z\frac{1}{b} \) are d-stable in \( M \) for \( \frac{1}{b} \in M \). Let \( N = Z\frac{1}{b} \), \( \alpha : M \to M/\langle N \rangle \), we will show first that \( \alpha(I) = 0 \) and then show that \( \alpha\left(\frac{1}{b}\right) = 0 \).

Assume that \( \alpha(I) = \frac{m}{n} + \langle N \rangle \) (note that \( \frac{m}{n} \in N \leftrightarrow n|\langle b \rangle \); also remember that both \( n \) and \( b \) are square free). Now \( \alpha(I) = \alpha\left(\frac{n}{n}\right) = n\alpha\left(\frac{1}{n}\right) \), if \( \alpha\left(\frac{1}{n}\right) = k + \langle N \rangle \), then \( \frac{nk}{1} + \langle N \rangle = \frac{m}{n} + \langle N \rangle \), hence \( \frac{m}{n} - \frac{nk}{1} = \frac{a}{b} \in N \), then \( \frac{mb - na}{n^2} = b\left(\frac{k}{1}\right) \). Since \( \frac{k}{1} \in M \), so \( \frac{mb - na}{n^2} \in M \) and we must have \( n|\langle b \rangle \), and then \( n|\langle b \rangle \) which implies \( \frac{m}{n} \in \langle N \rangle \), that is \( \alpha(I) = 0 \). Next, let \( \alpha\left(\frac{1}{b}\right) = \frac{p}{q} + \langle N \rangle \), we have \( 0 = \alpha\left(\frac{1}{b}\right) = b\langle\frac{p}{q}\rangle + \langle N \rangle \), hence \( \frac{bp}{q} \in N \), that is, \( \frac{bp}{q} \in \langle N \rangle \), which implies \( \frac{p}{q} = \frac{c}{b^2} \), but \( \frac{p}{q} \in M \), so \( b|c \) and \( \frac{p}{q} \in N \), that is, \( \alpha\left(\frac{1}{b}\right) = 0 \), in other words \( N \subseteq \ker(\alpha) \), hence \( N \) is d-stable. \( \diamondsuit \)

6. \( M \) has the qe-property and hence (by Corollary 2.7) is fully d-stable.

**Proof:** First note that if \( y = \frac{a}{b} \) and \( x = \frac{1}{b} \) are elements of \( M \) then \( M/Zx \) can be embedded in \( M/Zy \) by \( m + Zx \mapsto am + Zy \). Let \( x = \frac{1}{b} \) and \( b = p_1p_2...p_n \) for distinct primes \( p_1, p_2, ..., p_n \), let \( A \) be a submodule of \( M \) containing \( y \). Let \( N = \langle p_1, p_2, ..., p_n \rangle \), \( J = \langle p \in \text{PR} \ | 1/p \rangle \) is in the set of generators of \( A \), \( K = \text{PR} - J \) and \( L = \text{PR} - N \). It is clear that \( N \subseteq J \) and \( K \subseteq L \), also it is clear that \( M = A + B \), where \( B = \sum_{p \in K} Z\frac{1}{p} \) and note that \( A \cap B = Z \).

Now \( M/A \cong B/Z \cong \bigoplus_{p \in K} (Z/pZ) \). On the other hand \( Zx = \sum_{p \in N} Z\frac{1}{p} \). Hence
\[ \frac{M}{x} \cong \sum_{p \in \mathbb{L}} \frac{1}{p} \mathbb{Z} \cong \bigoplus_{p \in \mathbb{L}} (\mathbb{Z}/p\mathbb{Z}) \], then we conclude that \( M/A \) can be embedded in \( M/Zx \) (hence in \( M/Zy \), by the above note).

\[ \Diamond \]

**Example 2.12.** Let \( M = \mathbb{Z}[x] \), the ring of polynomials over \( \mathbb{Z} \), be considered as a module over itself, then \( M \) is a cyclic module and hence it satisfies the property of Theorem 2.8. Let \( N = \langle 2, x \rangle \) be the ideal of \( M \) generated by 2 and \( x \), it is known that \( N \) is a maximal (submodule) in \( M \) and hence \( M/N \) is simple, while \( M/\langle x \rangle \cong \mathbb{Z} \) which contain no simple submodule, that is, \( M/N \) cannot be embedded in \( M/\langle x \rangle \), so \( M \) does not satisfy the qe-property. Certainly, \( M \) is a fully d-stable module. \( \Diamond \)

In [3], two equivalent concepts were introduced and investigated, namely, fully pseudo d-stable and d-terse modules. The last one is: "a module is d-terse if it has no distinct isomorphic factors". An analogous necessary (but not sufficient) condition for principal d-stability is proved in the following.

**Proposition 2.13.** Let \( M \) be a principally d-stable module. If \( x, y \in M \) and \( M/Rx \cong M/Ry \), then \( Rx = Ry \).

**Proof:** Let \( \varphi : M/Rx \to M/Ry \) be an isomorphism, \( \pi_x \) and \( \pi_y \) be the natural epimorphisms onto \( M/Rx \) and \( M/Ry \) respectively, let \( \alpha = \varphi \circ \pi_x, \beta = \varphi^{-1} \circ \pi_y \), then (by hypothesis \( M \) is principally d-stable) we have \( Ry \subseteq \ker \alpha = \pi_x^{-1}(\ker \varphi) = Rx \) and \( Rx \subseteq \ker \beta = \pi_y^{-1}(\ker \varphi^{-1}) = Ry \). Therefore \( Rx = Ry \).

\( \Diamond \)

**Remark 2.14.** By the above Proposition we can deduce, simply, that the \( \mathbb{Z} \)-module \( Q \) (which is not fully d-stable, see [2]) is not principally d-stable too. Note that \( Q/Z \cong Q/Zx \), for each \( x \in Q \). Similarly the \( \mathbb{Z} \)-module \( Z/(p^n) \) is isomorphic to each of its factors, that is, any two factors of it are isomorphic, hence it is not principally d-stable.

In the following we will investigate the coincidence of principal d-stability with full d-stability over certain type of rings. First we need to recall some facts about duo and quasi-projective modules.

**Proposition 2.15.** [10] Let \( R \) be a Dedekind domain. Then the following statements are equivalent for a finitely generated \( R \)-module \( M \):

(i) \( M \) is a duo module.

(ii) \( M \cong I \) for some ideal \( I \) of \( R \) or \( M \cong (R/P_1^{n_1}) \oplus \ldots \oplus (R/P_k^{n_k}) \) for some positive integers. 

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\( k, n_1, \ldots, n_k \) and distinct maximal ideals \( P_i \) (1 \( \leq \) \( i \) \( \leq \) \( k \)) of \( R \).

Note that the first possibility of statement (ii) means \( M \) is torsion free and the second is torsion.

**Proposition 2.16.** [11] A torsion module \( M \) over a Dedekind domain \( R \) is quasi-projective if and only if each \( P \)-primary component \( M_P \) is a direct sum copies of the same cyclic module \( R/P^k \) for some fixed positive integer \( k \) depending on \( P \).

**Proposition 2.17.** [11] A torsion module \( M \) over a Dedekind domain \( R \) is quasi-projective if and only if \( M \) is quasi-injective but not injective.

Now we are ready to prove the following theorem which leads, further, to a link between the two dual concepts, full stability and full d-stability in certain conditions.

**Theorem 2.18.** Let \( R \) be a Dedekind domain. Then the following statements are equivalent for a finitely generated \( R \)-module \( M \):

(i) \( M \) is duo.

(ii) \( M \) is fully d-stable.

(iii) \( M \) is principally d-stable.

**Proof:** (i) \( \Rightarrow \) (ii). By Proposition 2.15, \( M \) is a duo module implies either \( M \cong I \) for some ideal \( I \) of \( R \) (which is projective, since every ideal of a Dedekind domain is projective [4], p.215) or \( M \cong (R/P_1^{n_1}) \oplus \ldots \oplus (R/P_k^{n_k}) \) for some positive integers \( k, n_1, \ldots, n_k \) and distinct maximal ideals \( P_i \) (1 \( \leq \) \( i \) \( \leq \) \( k \)) of \( R \) (which is quasi-projective by Proposition 2.16). In any case \( M \) is fully d-stable ([2], Proposition 2.3).

(ii) \( \Rightarrow \) (iii). Clear by definitions.

(iii) \( \Rightarrow \) (i). by Corollary 2.2. 

**Corollary 2.19.** For a finitely generated torsion module \( M \) over a Dedekind domain \( R \), the following statements are equivalent:

(i) \( M \) is fully stable.

(ii) \( M \) is fully d-stable.

**Proof:** \( M \) is fully stable implies \( M \) is duo, then by Proposition 2.10 and the note after it, we have \( M \cong (R/P_1^{n_1}) \oplus \ldots \oplus (R/P_k^{n_k}) \), which means that \( M \) is quasi-projective. Hence \( M \) is fully d-stable([2], Proposition 2.3).
Conversely, if $M$ is fully d-stable, then it is duo and hence quasi-projective (see part one). Now by Proposition 2.17, $M$ is quasi-injective. Therefore $M$ is fully stable (see [1]).

Remarks 2.20.

(i) $Z_{(p^n)}$ is a torsion module over a Dedekind domain, which is fully stable [1] but not fully d-stable [2]. Note that this module is not finitely generated.

(ii) $Z$ is a Dedekind domain, it is finitely generated module over itself, fully d-stable [2] but not fully stable [1]. It is clear that $Z$ is torsion free $Z$-module.

(iii) By the above theorem and a Corollary in [1], we can conclude the following statement: "A finitely generated torsion module $M$ over a Dedekind domain $R$ is fully d-stable if and only if, for each $x,y \in M$,

$ann_R(y) = ann_R(x)$ implies $Rx = Ry$.

We need to recall another fact about duo modules, in order to prove a next result.

Lemma 2.21. [10] Let $R$ be a domain. An $R$-module $M = M_1 \oplus M_2$, with a non zero torsion free submodule $M_1$, and a non zero submodule $M_2$, is not duo. \[\triangleleft\]

The proof of the following theorem can be found implicitly in the proof of Theorem 2.18, but we will give another proof.

Theorem 2.21. Let $M$ be a finitely generated module over a P. I. D., $R$. Then $M$ is principally d-stable if and only if it is fully d-stable.

Proof: Let $M$ be a finitely generated module over a P. I. D., $R$. It is known that $M = F \oplus T(M)$, where $F$ is a free module and $T(M)$ is the torsion submodule of $M$ (see, for example, [7]). We have the following cases:

(i) $T(M) = 0$, then $M$ is free, hence either $M \cong R$ which is fully d-stable, or $M \cong R \oplus \ldots \oplus R$, $k$ times and $k > 1$, which implies $M$ is not duo, so neither fully nor principally d-stable.

(ii) $F \neq 0$ and $T(M) \neq 0$, then by Lemma 2.21, $M$ is not duo, so neither fully nor principally d-stable. (note: it is known that any free module over a P. I. D. is torsion free)

(iii) $F = 0$, then $M$ is torsion, hence by the proof of Corollary 2.19 and that a principally d-stable module is duo, $M$ is fully d-stable if and only if $M$ is principally d-stable. (note that a P. I. D. is Dedekind domain) \[\triangleleft\]

Now we collect the cases and conditions that leads to the equivalence of the two concepts, full and principal d-stability, that we get (till now) by the following:

1. quasi-projective modules.
2. modules with q-e property.
3. finitely generated modules over Dedekind domain.
The following statement about principally d-stable modules, has an analogous statement in the case of fully d-stable which is proved in [2], but we will give a proof for completeness.

**Proposition 2.22.** If $M$ is a torsion free principally d-stable module over an integral domain $R$ which is not a field, then $M$ is not injective.

**Proof:** Assume $M$ is injective, then it is divisible. Let $0 \neq r$ be a non invertible element of $R$, then for each $x \in M$, there exists $y \in M$ such that $x = ry$. Define $f : M \to M$ by $f(x) = y \leftrightarrow x = ry$. $f$ is an endomorphism of $M$ (since $M$ is torsion free). $M$ is principally d-stable implies $M$ is duo (Corollary 2.2), hence for each $x \in M$, there exists $s \in R$ such that $f(x) = sx$ [10], so we have $rsx = x$ which implies $rs = 1$ (since $M$ is torsion free) and this contradicts the assumption that $r$ is not invertible. Therefore $M$ is not injective.

**Corollary 2.23.** Let $R$ be an integral domain, which is not a field, $M$ an injective principally d-stable module over $R$, then $M$ is not torsion free.

In the following we have another result about torsion free modules over integral domain. Recall that, in case of torsion free module $M$ the "rank" is the maximum number (cardinal number) of linearly independent elements in $M$ (see [6]).

**Proposition 2.24.** Let $M$ be a torsion free module over an integral domain $R$. If $M$ is quasi-injective of rank >1, then $M$ is not duo, consequently neither fully d-stable nor principally d-stable.

**Proof:** Assume that $x$, $y$ are two linearly independent elements in $M$, then $Rx \cap Ry = \emptyset$. Let $f : Rx \to M$ be defined by $f(rx) = ry$, then $f$ is an $R$-homomorphism, that can be extended to an endomorphism, say $g$, of $M$ (since $M$ is quasi-injective) and it is clear that $g(Rx) = Ry \subset Rx$, that is, $M$ is not duo.

In [3], we prove an equivalent statement to the definition of fully d-stable module which was "$M$ is fully d-stable if and only if $\ker g \subset \ker f$ for each $R$- module $A$ and any two $R$-homomorphisms $f, g : M \to A$ with $g$ surjective". In the end of this section a similar statement for principally d-stable module can be stated, and the proof will be omitted.

**Proposition 2.25.** Let $M$ be an $R$-module. $M$ is principally d-stable if and only if for each $R$- module $A$ and any two $R$-homomorphisms $f, g : M \to A$ with $g$ surjective and $\ker g$ is cyclic in $M$, $\ker g \subset \ker f$.

3- Full d-stability and Endomorphism ring

The endomorphism ring of a module, sometimes, gives additional information about the module itself, so it is natural to investigate the endomorphism ring of a fully d-stable module (and in particular principally d-stable module), to this aim we have the following results.

First recall the concept of "regular module", which is a generalization of the concept of Von Neumann's regular
ring, "there have been considered three types of modules by Fieldhouse, Ware and Zelmanowitz each called regular. The Fieldhouse-regular module was defined to be a module whose submodules are pure submodules and the Ware-regular module was defined as a projective module in which every submodule is a direct summand, while a left module $M$ over a ring $R$ is called a Zelmanowitz-regular module if for each $x \in M$ there is a homomorphism $f : M \to R$ such that $f(x)x = x$" [5]. Azumaya in [5], consider the following definition "a module $M$ is regular if every cyclic submodule is a direct summand". This definition is more convenience for our aim since the projectivity condition leads to the equivalence of the duo, fully d-stability and principal d-stability concepts, but we need to investigate the last two separately. So we will consider the Azumaya-regular definition:

**Definition 3.1.** [5] An $R$-module is regular if each of its cyclic submodule is a direct summand.

**Proposition 3.2.** If $M$ is a regular $R$-module and if $\text{End}_R(M)$ is commutative, then $M$ is a duo module.

**Proof:** Let $f \in \text{End}_R(M)$ and $x \in M$, since $M$ is regular, we have $M = Rx \oplus L$ for some submodule $L$ of $M$. Assume that $f(x) = rx + l$, $r \in R$ and $l \in L$. Let $\pi : M \to M$ defined by $\pi(sx + l) = sx$ for each $s \in R, t \in L$.

Now, $f(\pi(x)) = f(x) = rx + l$ and $\pi(f(x)) = rx$, but $\text{End}_R(M)$ is commutative, so, $f(x) = rx$. Therefore $M$ is a duo module. (lemma 1.1, [10])

**Corollary 3.3.** If $M$ is a regular $R$-module and if $\text{End}_R(M)$ is commutative, then $M$ is principally $d$-stable.

**Proof:** By proposition 3.2 $M$ is duo and by ([2], proposition 3.1) any direct summand of $M$ is $d$-stable, but $M$ is regular, hence any cyclic submodule is $d$-stable.

**Corollary 3.4.** If $M$ is a regular quasi-projective $R$-module and if $\text{End}_R(M)$ is commutative, then $M$ is fully $d$-stable.

**Lemma 3.5.** If $R$ is a commutative ring and $M$ is a duo $R$-module, then $\text{End}_R(M)$ is commutative.

**Proof:** Let $f, g \in \text{End}_R(M)$ and $x \in M$, then $f(x) = rx$ and $g(x) = sx$ for some $r, s \in R$ (lemma 1.1, [10]). Hence $f(g(x)) = f(sx) = sf(x) = srx$ and $g(f(x)) = g(rx) = rg(x) = rsx$, since $R$ is commutative, we have $f(g(x)) = g(f(x))$. Therefore $\text{End}_R(M)$ is commutative.

Recall that in [2], we show that "every quasi-projective duo $R$-module is fully $d$-stable. So we have the following result.

**Corollary 3.6.** If $R$ is a commutative ring, and $M$ is a regular quasi-projective $R$-module, then $M$ is fully $d$-stable if and only if, $\text{End}_R(M)$ is commutative.

**Proof:** (⇒) by lemma 3.5 and ([2], proposition 2.3).

(⇐) by proposition 3.2 and ([2], proposition 2.3).

**Corollary 3.7.** If $R$ is a commutative ring, and $M$ is a regular $R$-module, then $M$ is principally $d$-stable if and only if, $\text{End}_R(M)$ is commutative.

**Proof:** (⇒) by lemma 3.5 and corollary 2.2. (⇐) by corollary 3.3.
Corollary 3.8. If $R$ is a commutative ring and $M$ is a regular quasi-projective $R$–module, then $M$ is fully $d$-stable if and only if, $\text{End}_R(M)$ is fully $d$-stable.

Lemma 3.9. If $M$ is a regular $R$–module, $x \in M$ and $\alpha : M \rightarrow M/Rx$, then $\alpha$ can be lifted to an endomorphism of $M$.

Proof: Since $M$ is regular, $M = Rx \oplus L$, for some submodule $L$ of $M$, let $m \in M$, and assume that $\alpha(m) = rx + l$, $r \in R$ and $l \in L$. then we can write $\alpha(m) = l + Rx$, $l \in L$. also $l$ is unique for each $m \in M$ for if $l_1 + Rx = l_2 + Rx$, then $l_1 - l_2 \in R \cap L = 0$. Hence we can define $f : M \rightarrow M$ by $f(m) = l \Leftrightarrow \alpha(m) = l + Rx$, it clear that $\pi \circ f = \alpha$.

We can summarize the previous results in the following Corollary.

Corollary 3.10. If $R$ is a commutative ring, $M$ is a regular $R$–module, then the following statements are equivalent:

1. $M$ is principally $d$-stable.
2. $\text{End}_R(M)$ is a commutative ring.
3. $\text{End}_R(M)$ is fully $d$-stable.

A similar result is found in [1] but in place of statement 1 there was "$M$ is a fully stable module", from which we get a link between full stability and principal $d$-stability, that is,

Corollary 3.11. If $R$ is a commutative ring, $M$ is a regular $R$–module, then the following statements are equivalent:

1. $M$ is fully stable.
2. $M$ is principally $d$-stable.

Regularity of a module (in the mentioned sense) has other effect for $d$-stability (even stability), see the following.

Proposition 3.12. Let $M$ be a torsion free module over an integral module $R$. If $M$ is regular (but not simple), then it is not duo and consequently neither fully $d$-stable nor principally $d$-stable and not fully stable.

Proof: Let $0 \neq x \in M$ such that $M \neq Rx$ then $M = Rx \oplus N$ for some nonzero submodule $N$ of $M$, but $Rx$ is torsion free, so by Lemma 2.21 $M$ is not duo.

Other properties can be added for the endomorphism ring of a module, when it is hollow, (that is, the sum of any two proper submodules does not equal the module itself). Recall that an $R$–module $M$ is hopfian if every surjective endomorphism of $M$ is an isomorphism.

Proposition 3.13. If $M$ is a fully $d$-stable module over a commutative ring $R$, and if $M$ is hollow, then $\text{End}_R(M)$ is a commutative local ring.

Proof: Since $M$ is a fully $d$-stable, it is duo and hence by lemma 3.5 $\text{End}_R(M)$ is a commutative ring. Now $M$ is hopfian (see [2], Proposition 2.16), hence any non invertible element of $\text{End}_R(M)$ is not surjective. Let $L = \{ f \in \text{End}_R(M) : \text{Im} f \neq M \}$, $L$ is the subset of all non invertible elements of $\text{End}_R(M)$. If $f, g \in L$, then $\text{Im}(f + g) \subset (\text{Im} f) + (\text{Im} g) \neq M$ (since $M$ is hollow), hence $f + g \in L$, that is, $L$ is
additively closed, and $\text{End}_R(M)$ is local (see [6], 7.1.1 and 7.1.2).

Recall that, a module $M$ has the exchange property if for any index set $I$, whenever $M \oplus N = \bigoplus_{i \in I} A_i$, for modules $N$ and $A_i$, then $M \oplus N = M \oplus (\bigoplus_{i \in I} B_i)$ for submodules $B_i$ of $A_i$, $i \in I$ (see [9]). Also, it is known that "An indecomposable module has the exchange property if and only if its endomorphism ring is local" (see [12]). Using this remark, proposition 3.10 and the fact that hollow module is indecomposable, we have the following:

**Corollary 3.14.** A fully $d$-stable hollow module has the exchange property.

R.B. Warfield proved the following: Let $M$ be a module with a local endomorphism ring and suppose $A$ and $B$ are modules such that $A \oplus M \cong B \oplus M$, then $A \cong B$ (see [12]).

Hence we can add the following corollary:

**Corollary 3.15.** A fully $d$-stable hollow module has the cancellation property.

**REFERENCES**