Weak Forms of $\omega$-Open Sets in Bitopological Spaces and Connectedness

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Abstract

The aim of this paper is to introduce a new classes of weak $\omega$-open sets in bitopological spaces then study the relations between those classes and some properties. Other aim is to introduce certain type of connectedness in bitopological spaces relative to the new classes of sets introduced in the first part, and get some results.

Keywords: $\omega$ pre open set, pre $\omega$ open set, $\omega$ semi open set, semi $\omega$ open set, $\omega$ $\alpha$ open set, $\alpha$ $\omega$ open set, $\omega$ $\beta$ open set, $\beta$ $\omega$ open set, $\omega$ b open set, b $\omega$ open set.

1. Introduction

The concepts of pre open sets, semi open sets, $\alpha$ open sets, $\beta$ open sets, and b-open sets introduced by many authors in topological spaces (cf. [2, 4, 6, 8, 10]) and extended to bitopological spaces by others (cf. [9, 11]). The concept of $\omega$-open sets was introduced and studied by many authors (cf. [3,12]), and extended to bitopological spaces in [5], by defining the concept of $\tau_1 \tau_2$–generalized $\omega$-closed set.

In this paper many types of weak open sets in bitopological spaces will be defined, Relations between those sets will be discussed, properties such as supra and infra topological structures will be determined.

Also a new type of connectedness for bitopological spaces will be defined and preserving that type of connectedness under certain type of map between bitopological spaces will be proved, many other results and counter examples also will be showed.

Throughout this paper the following notation will be used: $\subset$ denotes subset (not necessarily proper), $A^c$ denotes the complement of $A$ in the space (that $A$ is subset of).

If $(X, \tau_1, \tau_2)$ is a bitopological space, $A \subset X$, $i$-int $A$ and $j$-cl $A$ denote the interior and closure of $A$ relative to $\tau_i$ and $\tau_j$ respectively, $i$-open(closed) set denotes $\tau_i$ open(closed) set ($i,j \in \{1,2\}$).

1.1. Definition [4, 11]

Let $(X, \tau_1, \tau_2)$ be a bitopological space, $A \subset X$, $A$ is said to be:

(i) $ij$- p open set if $A \subset i$-int ($j$-cl $A$).
(ii) $ij$- s open set if $A \subset j$-cl($i$-int $A$).
(iii) $ij$- $\alpha$ open set if $A \subset i$-int($j$-cl($i$-int $A$)).
(iv) $ij$- $\beta$ open set if $A \subset j$-cl($i$-int ($j$-cl $A$)).
1.2. Remark

It is clear from definition that in any bitopological space the following hold:

(i) Every i-open set is ij- p open, ij- s open, ij- α open, ij- β open and ij- b open set.
(ii) Every ij- p open set is ij- β open.
(iii) Every ij- α open set is ij- s open.
(iv) Every ij- p open(ij- s open) set is ij- b open set.
(v) The concepts of ij- p open and ij- s open sets are independent.
(vi) The concepts of ij- α open and ij- β open sets are independent.

2. Weak Forms of ω-Open Sets in Bitopological Spaces

First recall the following definition from topological spaces.

2.1. Definition [3]

Let $ (X, \tau) $ be a topological space, $ A \subset X $, $ x \in X $ is called a condensation point if for each $ U \in \tau $ with $ x \in U $, the set $ U \cap A $ is uncountable. A is said to be an $ \omega $-closed if it contains all its condensation points. The complement of $ \omega $-closed set is said to be $ \omega $-open set. Equivalently a set $ W $ is $ \omega $-open if for each $ x \in W $, there exist $ U \in \tau $ with $ x \in U $ and $ U-W $ is countable.

The family of all $ \omega $-open sets in $ (X, \tau) $, denoted by $ \tau_\omega $, forms a topology on $ X $ finer than $ \tau $. The $ \omega $-closure and $ \omega $-interior of a set $ A $, will be denoted by $ \text{cl}_\omega A $ and $ \text{int}_\omega A $ resp., are defined by:

\[
\text{cl}_\omega A = \cap \{F \subset X | F \text{ is } \omega \text{-closed and } A \subset F \}
\]

\[
\text{int}_\omega A = \cup \{G \subset X | G \text{ is } \omega \text{-open and } G \subset A \}
\]

2.2. Note

The following notes are clear:

(i) If $ (X, \tau) $ is any topological space, $ A \subset X $, then $ \text{int} A \subset \text{int}_\omega A $ and $ \text{cl}_\omega A \subset \text{cl} A $.

(ii) If $ X $ is a countable set and $ \tau $ is any topology on $ X $, then all the subsets of $ X $ are $ \omega $-closed and $ \omega $-open, i.e., $ \tau_\omega = \text{P}(X) $.

In what follows, let $ i,j \in \{1,2\} $ and $ i \neq j $.

2.3. Definition

Let $ (X, \tau_1, \tau_2) $ be a bitopological space, $ A \subset X $. A is said to be:

ij- $ \omega $ pre open, if $ A \subset \text{int}_\omega (j-\text{cl} A) $.
ij- $ \omega $ open, if $ A \subset \text{int}_\omega (j-\text{cl} A) $.
ij- $ \omega $ semi open, if $ A \subset j-\text{cl}_\omega (i-\text{int} A) $.
ij- $ \omega $ open, if $ A \subset i-\text{int}_\omega (j-\text{cl}(i-\text{int} A)) $.
ij- $ \alpha $ open, if $ A \subset i-\text{int}_\omega (j-\text{cl}(i-\text{int} A)) $.
ij- $ \beta $ open, if $ A \subset j-\text{cl}_\omega (i-\text{int}(j-\text{cl} A)) $.
ij- $ \omega $ open, if $ A \subset j-\text{cl}_\omega (i-\text{int}(j-\text{cl} A)) $.
ij- $ \omega $ semi open, if $ A \subset j-\text{cl}_\omega (i-\text{int}(j-\text{cl} A)) $.
ij- $ \omega $ pre open, if $ A \subset \text{int}_\omega (j-\text{cl} A) \cup j-\text{cl}_\omega (i-\text{int} A) $.
ij- $ \omega $ open, if $ A \subset \text{int}_\omega (j-\text{cl} A) \cup j-\text{cl}_\omega (i-\text{int} A) $.

The set ij-\$ \omega \$ pre open (ij-pre $ \omega \$ open; ij-\$ \omega \$ semi open; ij-semi $ \omega \$ open) will be denoted briefly ij-\$ \omega \$ p open (ij-\$ \omega \$ p open; ij- $ \omega \$ s open; ij- s $ \omega \$ open).
2.4. Remark

If $(X, \tau_1, \tau_2)$ is a bitopological space, $A$ is a countable subset of $X$, then:

(i) $A$ is $ij$-$p\omega$ ($ij$-$s\omega$; $ij$-$\alpha\omega$; $ij$-$\omega\beta$) open if and only if it is $i$-open.

(ii) $A$ is $ij$-$\omega b$ ($ij$-$b\omega$) open, if it is $i$-open.

(iii) $A$ is $ij$-$\omega b$ ($ij$-$b\omega$) open implies it is $ij$-$\omega p$ ($ij$-$s\omega$) open.

Proof: (i) and (ii) since $cl_{\omega}A = A$ (relative to any topology), when $A$ is countable.

(iii) since $cl_{\omega}A = A$ (when $A$ is countable) and $int_{\omega}A \subset int_{\omega}A$ in general.

2.5. Remark

If $(X, \tau_1, \tau_2)$ is a bitopological space, and if $A$ is a subset of $X$ such that $A^c$ is countable, then $A$ is $ij$-$\omega h$ open for $h=p, \alpha, s$, and $ij$-$\omega b$ ($ij$-$b\omega$) open, if $h=s, \beta, b$.

Proof: if $A^c$ is countable then $(j-cl A)^c$ is countable too, which implies that $i-int_{\omega}(j-cl A) = j-cl A$ and $j-cl(i-int_{\omega}A) = j-cl A$. Now the fact that $A \subset j-cl A$ completes the proof.

2.6. Remark

If $X$ is countable, $\tau_1$ and $\tau_2$ are any two topologies on $X$, $A \subset X$, then:

(i) $A$ is $ij$-$\omega h$ open for $h=p, \alpha, b$ and $ij$-$\omega h$ for $h=s$, $\beta, b$.

(ii) $A$ is $ij$-$p\omega$ ($ij$-$\alpha\omega$; $ii$-$s\omega$; $ij$-$\omega\beta$) open if and only if it is $i$-open.

Proof: (i) since $cl_{\omega}A = A$ and $int_{\omega}A = A$ (when $X$ is countable).

(ii) By 2.4 and 2.5.

2.7. Theorem

If $X$ is a countable set, $\tau_1$ and $\tau_2$ are any two topologies on $X$, then; the family of all $ij$-$\omega h$ ($ij$-$h\omega$) open subsets of $X$, $h=p, \alpha, s, \beta, b$, form a topology on $X$.

Proof: By 2.6:

(i) It is $P(X)$ for the cases $ij$-$\omega h$ with $h=p, \alpha, b$ and the cases $ij$-$\omega h$ with $h=s, \beta, b$.

(ii) It is $\tau_i$ for the cases $ij$-$p\omega$, $ij$-$\alpha\omega$, $ij$-$\omega s$ and $ij$-$\omega\beta$.

2.8. Remark

Let $(X, \tau_1, \tau_2)$ be a bitopological space, the following relations between the sets defined in 2.3 hold:

(i) Every $ij$-$h\omega$ open is $ij$-$h$ open set ($h=p, \alpha$) but not the converse.

(ii) Every $ij$-$\omega h$ open is $ij$-$h$ open set ($h=s, \beta$) but not the converse.

(iii) Every $ij$-$h$ open is $ij$-$\omega h$ open set ($h=p, \alpha$) but not the converse.

(iv) Every $ij$-$h$ open is $ij$-$\omega h$ open set ($h=s, \beta$) but not the converse.

Proof: straightforward by definitions.

2.9. Examples

The following examples show that the converse in the previous remark are not true:

Let $X=\{a, b, c, d\}$; $\tau_1=\tau_2=\{\varnothing, \{a\}, \{a, b\}, \{a, b, c\}\}$:

(i) $A=\{a, d\}$ is 12-$p$ but not 12-$p\omega$ open set.

$B=\{a, b, d\}$ is 12-$\alpha$ but not 12-$\alpha\omega$ open set.

(ii) $A$ is 12-$s$ but not 12-$\omega s$ open set.

$A$ is 12-$\beta$ but not 12-$\omega \beta$ open set.

(iii) $C=\{b, c\}$ is 12-$\omega p$ but not 12-$p$ open set.

$D=\{c, d\}$ is 12-$\omega \alpha$ but not 12-$\alpha$ open set.

(iv) $C$ is 12-$s\omega$ but not 12-$s$ open set.

$C$ is 12-$\beta\omega$ but not 12-$\beta$ open set.
2.10. Remark
Let \((X, \tau_1, \tau_2)\) be a bitopological space, then every \(ij-\alpha\omega\) open set is \(ij-\beta\omega\) open set but not the converse, where the following pairs of concepts are independent:

(i) \(ij-\alpha\omega\) open and \(ij-\omega\) open.
(ii) \(ij-\alpha\omega\) open and \(ij-\beta\omega\) open.
(iii) \(ij-\alpha\omega\) open and \(ij-\beta\omega\) open.

Proof: \(i\)-int \(A \subseteq A\) implies \(j\)-cl \((i\)-int \(A) \subseteq j\)-cl \(A\), so \(j\)-cl\(_\omega\)(\(i\)-int \(A)) \(\subseteq j\)-cl \(A\) and \(i\)-int\(_\omega\)(\(j\)-cl \(A)) \(\subseteq i\)-int \(j\)-cl \(A\). Hence an \(ij-\alpha\omega\) open set is an \(ij-\beta\omega\) open set.

Before giving examples to verify the other parts of the remark, the following note is needed.

2.11. Note
If \(X\) is uncountable, \(\tau = \{\varnothing, X\}\), \(A\) is an uncountable subset of \(X\), we have:

\[
\text{int}_\omega A = \begin{cases} \varnothing & \text{if } X - A \text{ is uncountable} \\ A & \text{if } X - A \text{ is countable} \end{cases}
\]

\[
\text{cl}_\omega A = \begin{cases} \varnothing & \text{if } A \text{ is uncountable} \\ X & \text{if } A \text{ is countable} \end{cases}
\]

2.12. Examples
Let \(X\) be uncountable, \(A\) an uncountable subset of \(X\) and \(B\) a countable subset of \(A\).

Suppose that \(\tau_1 = \{\varnothing, X\}, \tau_2 = \{\varnothing, X, A, B\}\), then the set \(A-B\) is:

(i) \(12-\alpha\omega\) open but not \(12-\omega\) open set.
(ii) \(12-\beta\omega\) open but not \(12-\omega\) open set (if \(A^c\) is countable).
(iii) \(12-\alpha\omega\) open but not \(12-\beta\omega\) open set (if \(A^c\) is countable).
(iv) \(12-\beta\omega\) open but not \(12-\alpha\omega\) open set.
(v) \(12-\alpha\omega\) open but not \(12-\beta\omega\) open (if \(A^c\) is uncountable).

Now take \(X = A \cup B \cup C \cup D\), where \(A, B, C, D\) are pair wise disjoint uncountable sets and suppose that \(\tau_1 = \tau_2 = \{\varnothing, X, A, B, A \cup B, A \cup B \cup C\}\), then the set \(A \cup C\) is:

(vi) \(12-\omega\) open but not \(12-\alpha\omega\) open set.
(vii) \(12-\beta\omega\) open but not \(12-\omega\) open set.

2.13. Remark
Let \((X, \tau_1, \tau_2)\) be a bitopological space, then the following relations hold:

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
ij-\alpha\omega\text{ open} & \rightarrow & ij-\beta\omega\text{ open} & \leftarrow \ij-\alpha\omega\text{ open} \\
4 & 5 & 6 & 7 \\
ij-\beta\omega\text{ open} & \leftarrow \ij-\beta\omega\text{ open} & \rightarrow \ij-\beta\omega\text{ open} & \leftarrow \ij-\omega\text{ open} \\
8 & 9 & 10 & 11 \\
ij-\alpha\omega\text{ open} & \rightarrow \ij-\beta\omega\text{ open} & \leftarrow \ij-\alpha\omega\text{ open} & \rightarrow \ij-\beta\omega\text{ open} \\
12 & 13 & 14 & 15 \\
ij-\beta\omega\text{ open} & \leftarrow \ij-\beta\omega\text{ open} & \rightarrow \ij-\beta\omega\text{ open} & \leftarrow \ij-\beta\omega\text{ open} \\
\end{array}
\]

The above arrows are not reversible, the set \(A-B\) in the previous example is a counter example for the arrows 1, 3, 5, and 6 (with \(A^c\) countable), for 7 and 8 (with \(A^c\) uncountable); the set \(A^c\) (with \(i=2\) and \(j=1\)) for the arrow 4 and finally the set \(A \cup C\) of the second part of the example for the arrow 2, verifying this fact.
2.14. Remark
Let \((X, \tau_1, \tau_2)\) be a bitopological space, then the concepts \(ij\)-\(b\) open; \(ij\)-\(\omega\) \(b\) open; \(ij\)-\(b\) \(\omega\) open are pairwise independent as shown in the following examples.

2.15. Example
(i) The set \(D\) in example 2.9 (iii) is \(12\)-\(\omega\) \(b\) open and \(12\)-\(b\) open but not \(12\)-\(b\) open set.
(ii) If \(X=\mathbb{R}, \tau_1=\tau_2=\) the usual topology, then \(Q\) is \(12\)-\(b\) open but not \(12\)-\(\omega\) \(b\) open.
(iii) If \(\{A, B, C, D\}\) is a partition of a set \(X\), where \(A\), \(C\) are countable and \(B\), \(D\) are uncountable subsets of \(X\) and if \(\tau_1=\tau_2=\{\emptyset, X, A, B, A \cup B, A \cup B \cup C\}\), then \(A \cup C\) is \(12\)-\(b\) open set but not \(12\)-\(\omega\) \(b\) open set.

The following diagram summarizes the results of 2.8, 2.10, 2.13 and 2.14:

\[
\begin{align*}
\text{ij-} \alpha \omega & \text{ open} \\
\text{ij-} p \omega & \text{ open} \\
\text{ij-} b \omega & \text{ open} \\
\text{ij-} s \omega & \text{ open} \\
\text{ij-} \beta \omega & \text{ open}
\end{align*}
\quad
\begin{align*}
\text{ij-} \alpha & \text{ open} \\
\text{ij-} p & \text{ open} \\
\text{ij-} b & \text{ open} \\
\text{ij-} s & \text{ open} \\
\text{ij-} \beta & \text{ open}
\end{align*}
\quad
\begin{align*}
\text{ij-} \omega \alpha & \text{ open} \\
\text{ij-} \omega p & \text{ open} \\
\text{ij-} \omega b & \text{ open} \\
\text{ij-} \omega s & \text{ open} \\
\text{ij-} \omega \beta & \text{ open}
\end{align*}
\]

Where \(\longrightarrow\) means the arrow is not reversible and \(\leftrightarrow\) means independent.

2.16. Definition
If \((X, \tau_1, \tau_2)\) is a bitopological space, a subset \(A\) of \(X\) is said to be \(ij\)-\(\omega\) \(h\) closed (\(ij\)-\(h\) \(\omega\) closed) set if its complement is \(ij\)-\(\omega\) \(h\) open (\(ij\)-\(h\) \(\omega\) open) where \(h=\ p, s, \alpha, \beta, b\).

Notation
Let \((X, \tau_1, \tau_2)\) be a bitopological space, the family of all \(ij\)-\(\omega\) \(h\) open (\(ij\)-\(h\) \(\omega\) open; \(ij\)-\(h\) \(\omega\) closed; \(ij\)-\(h\) \(\omega\) closed) subsets of \(X\) will be denoted by \(ij\)-\(\omega\) \(h\)\(O\)(\(X\)) (\(ij\)-\(h\) \(\omega\) \(O\)(\(X\)); \(ij\)-\(\omega\) \(h\)\(C\)(\(X\)); \(ij\)-\(h\) \(\omega\) \(C\)(\(X\))), where \(h=\ p, s, \alpha, \beta, b\).

Recall, a family \(\mu\) of subsets of \(X\) is called supra topological structures on \(X\) if \(\mu\) contains \(X, \emptyset\) and is closed under arbitrary union. And it is called infra topological structures on \(X\) if \(\mu\) contains \(X, \emptyset\) and is closed under finite intersections \([7]\).

2.17. Theorem
Let \((X, \tau_1, \tau_2)\) be a bitopological space, \(ij\)-\(\omega\) \(h\)\(O\)(\(X\)) and \(ij\)-\(h\) \(\omega\) \(O\)(\(X\)) where \(h=\ p, s, \alpha, \beta, b\) are supra topological structures on \(X\).

Proof: Since \(\text{int} \cup A_\lambda=\cup \text{int} A_\lambda\) and \(\cup \text{cl} A_\lambda \subset \text{cl} \cup A_\lambda\), the proof is straightforward.

2.18. Definition
A new closure operations on a subset \(A\) of a bitopological space \((X, \tau_1, \tau_2)\) can be defined as follows:
2.19. Remark

Let \((X, \tau_1, \tau_2)\) be a bitopological space.

(i) If \(A\) and \(B\) are two disjoint \(ij-\omega h\) open subsets of \(X\), then
\[ A \cap ij-\omega h \text{cl}(B) = B \cap ij-\omega h \text{cl}(A) = \varnothing. \]

(ii) If \(A\) and \(B\) are two disjoint \(ij-h\) open subsets of \(X\), then
\[ A \cap ij-\omega cl(B) = B \cap ij-h \omega cl(A) = \varnothing. \]

Proof: (i) will be proved, and the proof of (ii) is similar.

3. \(ij-\omega h\) (\(ij-h\)\) \(\omega\)) Connectivity

3.1. Definition

A bitopological space \((X, \tau_1, \tau_2)\) is said to be \(ij-\omega h\) disconnected (\(ij-h\) \(\omega\) disconnected) if it has a subset (other than \(\varnothing\) and \(X\)) which is both \(ij-\omega h\) open and \(ij-\omega h\) closed (\(ij-h\) \(\omega\) open and \(ij-h\) \(\omega\) closed), otherwise it is called \(ij-\omega h\) connected (\(ij-h\) \(\omega\) connected), \(h = p, s, \alpha, \beta, b\).

3.2. Theorem

If \(X\) is a countable set (with more than one point), \(\tau_1, \tau_2\) are any two topologies on \(X\), then \((X, \tau_1, \tau_2)\) is:

(i) \(ij-\omega h\) disconnected, for \(h = p, \alpha, b\).

(ii) \(ij-h\) \(\omega\) disconnected, for \(h = s, \beta, b\).

(iii) \(ij-\omega\) \((ij-\alpha, ij-\omega s, ij-\omega \beta)\) disconnected if and only if \((X, \tau_1)\) is disconnected.

Proof: By 2.6.

3.3. Remark

In general, \(i-\) disconnectivity implies \(ij-\omega h\) (\(ij-h\) \(\omega\) disconnectivity) \(h = p, s, \alpha, \beta, b\). But not the converse.

3.4. Example

(i) \((X, \tau_1, \tau_2)\) in the first part of example 2.12 is \(12-\omega\) disconnected (since \(A-B\) is both \(12-\omega\) open and \(12-\omega\) closed), where \((X, \tau_1)\) is connected.

(ii) \((X, \tau_1, \tau_2)\) in the second part of example 2.12 is \(12-\omega\) \(s\) disconnected.

(iii) and \(12-\omega\) \(\beta\) disconnected, since \(A \cup C\) is both \(12-\omega\) \(s\) (\(12-\omega\) \(\beta\)) open and \(12-\omega\) \(s\) (\(12-\omega\) \(\beta\)) closed, where \((X, \tau_1)\) is connected.

(iv) Let \(X = R, \tau_1=\) the usual topology and \(\tau_2=\{\varnothing, X\}\), then \((X, \tau_1, \tau_2)\) is \(12-\alpha\) \(\omega\) disconnected, since the set \((0,1)\) is both \(12-\alpha\) \(\omega\) open and \(12-\alpha\) \(\omega\) closed (many other sets exist), where \((X, \tau_1)\) is connected.

(v) Finally \((X, \tau_1, \tau_2)\) of example 2.9 is \(12-\omega h\) disconnected (\(h = s, \beta, b\)) and \(12-\omega h\) disconnected (\(h = p, \alpha, b\)), where \((X, \tau_1)\) is connected.
3.5. Definition

Let \((X, \tau_1, \tau_2)\) be a bitopological space, \(Y \subseteq X\), \(Y\) is said to be \(ij\)- connected if there exist two nonempty \(ij\)- open subsets of \(X\) \(G\) and \(H\) such that \(Y \subseteq G \cup H\), \(G \cap H = \emptyset\), \(Y \cap G \neq \emptyset\), and \(Y \cap H \neq \emptyset\).

 Otherwise \(Y\) is called \(ij\)- connected.

\[(h= p, s, \alpha, \beta, b)\]

3.6. Lemma

Let \((X, \tau_1, \tau_2)\) be a bitopological space, \(Y \subseteq X\). If \(Y\) is \(ij\)- connected and \(G\) and \(H\) are two \(ij\)- open subsets of \(X\) such that \(G \cap H = \emptyset\) and \(Y \subseteq G \cup H\), then either \(Y \subseteq G\) or \(Y \subseteq H\).

\[(h= p, s, \alpha, \beta, b)\]

**Proof:** Obvious by Definition 3.5.

3.7. Theorem

If \((X, \tau_1, \tau_2)\) is a bitopological space, \(\{Y_\lambda\}\) a family of \(ij\)- connected subsets of \(X\) that have a point in common, then \(\bigcup Y_\lambda\) is \(ij\)- connected.

\[(h= p, s, \alpha, \beta, b)\]

**Proof:** Assume that \(\{Y_\lambda\}\) is \(ij\)- connected sets but \(\bigcup Y_\lambda\) is \(ij\)- connected, let \(G\) and \(H\) be disjoint \(ij\)- open subsets of \(X\) with \(\bigcup Y_\lambda \subseteq G \cup H\), \((\bigcup Y_\lambda) \cap G \neq \emptyset\) and \((\bigcup Y_\lambda) \cap H \neq \emptyset\). Let \(x \in \bigcap Y_\lambda\), then either \(x \in G\) or \(x \in H\) (since \(G\) and \(H\) are disjoint). Now by Lemma 3.6 each \(Y_\lambda\) either subset of \(G\) or subset of \(H\) and since \(x \in Y_\lambda\) for each \(\lambda\) then either \(Y_\lambda \subseteq G\) for each \(\lambda\) or \(Y_\lambda \subseteq H\) for each \(\lambda\), that is, either \(\bigcup Y_\lambda \subseteq G\) or \(\bigcup Y_\lambda \subseteq H\) (a contradiction with the assumption, \((\bigcup Y_\lambda) \cap G \neq \emptyset\) and \((\bigcup Y_\lambda) \cap H \neq \emptyset\)). Hence \(\bigcup Y_\lambda\) is \(ij\)- connected.

The proof of the other case is similar.

3.8. Theorem

Let \((X, \tau_1, \tau_2)\) be a bitopological space, \(Y\) and \(Z\) \(ij\)- connected subset of \(X\). If \(Y \subseteq Z \subseteq \bigcup Y_\lambda\) \(ij\)- connected, then \(Z\) is \(ij\)- connected.

**Proof:** Assume that \(Z \subseteq \bigcup Y_\lambda\), where \(G\) and \(H\) are disjoint \(ij\)- open subsets of \(X\), \(Y \subseteq Z\) implies \(Y \subseteq G \cup H\), and by 3.6 either \(Y \subseteq G\) or \(Y \subseteq H\).

If \(Y \subseteq G\), then \(ij\)- open \(h\) \(cl Y \subseteq \bigcup Y_\lambda \subseteq \bigcup Y_\lambda \subseteq \bigcup Y_\lambda \subseteq clG\), so \(Z \subseteq \bigcup Y_\lambda \subseteq clG\), but by 2.19 \(H \cap \bigcup Y_\lambda \subseteq \bigcup Y_\lambda \subseteq clG = \emptyset\), hence \(Z \subseteq clG = \emptyset\).

Similarly, \(Y \subseteq H\) implies \(Z \subseteq clG = \emptyset\). Therefore \(Z\) is \(ij\)- connected.

3.9. Theorem

Let \((X, \tau_1, \tau_2)\) be a bitopological space, \(Y\) and \(Z\) \(ij\)- connected subset of \(X\). If \(Y \subseteq Z \subseteq \bigcup Y_\lambda\) \(ij\)- connected, then \(Z\) is \(ij\)- connected.

**Proof:** is similar.

3.10. Remark

Let \((X, \tau_1, \tau_2)\) be a bitopological space, if \((X, \tau_i)\) is disconnected, then \((X, \tau_1, \tau_2)\) is \(ij\)- disconnected.

**Proof:** Obvious since every \(i\)-open set is \(ij\)- open set, \((h= p, s, \alpha, \beta, b)\).

In what remaining of this section, the question about preserving \(ij\)- connectivity between two bitopological spaces under certain maps will be discussed.

First some notation is needed; let \((X, \tau_1, \tau_2), (Y, \sigma_1, \sigma_2)\) be two bitopological spaces and \(f: X \rightarrow Y\). \(f\) is \(i\)-continuous means \(f: (X, \tau_i) \rightarrow (Y, \sigma_i)\) is continuous, and \(f\) is \(j\)-open means \(f: (X, \tau_j) \rightarrow (Y, \sigma_j)\) is open. \(i, j \in \{1, 2\}\).

Also the following lemmas are needed:
3.11. Lemma
If \( f: (X, \tau) \rightarrow (Y, \sigma) \) is continuous and injective, \( B \subset Y \), then \( f^{-1}(\text{int}_{\omega}(B)) \subset \text{int}_{\omega}(f^{-1}(B)) \).

**Proof:** \( x \in f^{-1}(\text{int}_{\omega}(B)) \) implies \( f(x) \in \text{int}_{\omega}(B) \), that is, there is \( V \in \sigma \) such that \( f(x) \in V \) and \( V \cap B^c \) is countable, hence \( x \in f^{-1}(V) \in \tau \), and \( f^{-1}(V \cap B^c) = f^{-1}(V) \cap (f^{-1}(B))^c \) is countable (since \( f \) is injective), therefore \( x \in \text{int}_{\omega}(f^{-1}(B)) \).

**Note:** if \( f \) is continuous, it is clear that, \( f^{-1}(\text{int}(B)) \subset \text{int}(f^{-1}(B)) \).

3.12. Lemma
If \( f: (X, \tau) \rightarrow (Y, \sigma) \) is open and bijective, \( B \subset Y \), then \( f^{-1}(\text{cl}_{\omega}(B)) \subset \text{cl}_{\omega}(f^{-1}(B)) \).

**Proof:** \( x \notin \text{cl}_{\omega}(f^{-1}(B)) \) implies there is \( U \in \tau \), \( x \in U \) and \( U \cap f^{-1}(B) \) is countable, hence \( f(U) \cap B \) is countable (since \( f \) is bijective), but \( f(x) \in f(U) \in \sigma \) (since \( f \) is open). Therefore \( f(x) \notin \text{cl}_{\omega}(B) \), so \( x \notin f^{-1}(\text{cl}_{\omega}(B)) \).

**Note:** if \( f \) is open, it is clear that, \( f^{-1}(\text{cl}(B)) \subset \text{cl}(f^{-1}(B)) \).

3.13. Lemma
Let \( (X, \tau_1, \tau_2) \), \( (Y, \sigma_1, \sigma_2) \) be two bitopological spaces, \( f: X \rightarrow Y \) bijective, \( i \)-continuous and \( j \)-open. If \( B \) is an \( ij-\omega \) \( p \) open subset of \( Y \) then \( f^{-1}(B) \) is \( ij-\omega \) \( p \) open subset of \( X \).

**Proof:** \( B \subset i-\text{int}_{\omega}(j-\text{cl}(B)) \) implies \( f^{-1}(B) \subset f^{-1}(i-\text{int}_{\omega}(j-\text{cl}(B))), \) by 3.11 \( f^{-1}(i-\text{int}_{\omega}(j-\text{cl}(B))) \subset \text{int}_{\omega}(f^{-1}(j-\text{cl}(B))), \) since \( f^{-1}(j-\text{cl}(B)) \subset j-\text{cl}(f^{-1}(B)) \), hence, \( f^{-1}(B) \subset f^{-1}(i-\text{int}_{\omega}(j-\text{cl}(B))) \subset \text{int}_{\omega}(j-\text{cl}(f^{-1}(B))), \) Therefore \( f^{-1}(B) \) is \( ij-\omega \) \( p \) open subset of \( X \).

**Note:** \( ij-\omega \) \( p \) in the above lemma can be replaced by \( ij-\omega \) \( h \) \( (ij-\omega \) \( h) \), \( h= p, s, \alpha, \beta, b \), and the proof will stay similar.

3.14. Theorem
Let \( (X, \tau_1, \tau_2) \), \( (Y, \sigma_1, \sigma_2) \) be two bitopological spaces, \( f: X \rightarrow Y \) bijective, \( i \)-continuous and \( j \)-open.

If \( (X, \tau_1, \tau_2) \) is \( ij-\omega \) \( h \) \( (ij-\omega \) \( h) \) connected then \( (Y, \sigma_1, \sigma_2) \) is \( ij-\omega \) \( h \) \( (ij-\omega \) \( h) \) connected too. ( \( h= p, s, \alpha, \beta, b \))

**Proof:** One case will be proved, other cases are similar.

Assume that \( (X, \tau_1, \tau_2) \) is \( ij-\omega \) \( p \) connected, \( f \) is bijective, \( i \)-continuous and \( j \)-open, and assume that \( (Y, \sigma_1, \sigma_2) \) is \( ij-\omega \) \( p \) disconnected. Let \( A \) and \( B \) be two non empty disjoint \( ij-\omega \) \( p \) open subsets of \( Y \) such that \( Y=A \cup B \). By 3.13 \( f^{-1}(A) \) and \( f^{-1}(B) \) are \( ij-\omega \) \( p \) open subsets of \( X \), they are nonempty and disjoint (since \( f \) is bijective) and \( X=f^{-1}(A) \cup f^{-1}(B) \), which contradicts the assumption that \( (X, \tau_1, \tau_2) \) is \( ij-\omega \) \( p \) connected. Therefore \( (Y, \sigma_1, \sigma_2) \) is \( ij-\omega \) \( p \) connected.

3.15. Lemma
If \( f: (X, \tau) \rightarrow (Y, \sigma) \) is open and bijective, \( A \subset X \), then \( f(\text{int}_{\omega}(A)) \subset \text{int}_{\omega}(f(A)) \).

**Proof:** \( y \in f(\text{int}_{\omega}(A)) \) implies \( y=f(x) \), \( x \in \text{int}_{\omega}(A) \), so there is \( U \in \tau \), \( x \in U \) such that \( U \cap A^c \) is countable, hence \( f(U) \cap (f(A))^c \) is countable (since \( f \) is bijective), where \( f(U) \in \sigma \) (since \( f \) is open), and \( y \in f(U) \).

Therefore \( y \in \text{int}_{\omega}(f(A)) \).

**Note:** if \( f \) is open , it is clear that, \( f(\text{int}(A)) \subset \text{int}(f(A)) \).

3.16. Lemma
If \( f: (X, \tau) \rightarrow (Y, \sigma) \) is continuous and bijective, \( A \subset X \), then \( f(\text{cl}_{\omega}(A)) \subset \text{cl}_{\omega}(f(A)) \).

**Proof:** \( y \notin \text{cl}_{\omega}(f(A)) \) implies there is \( V \in \sigma \), \( y \in V \) and \( V \cap f(A) \) is countable, which implies \( f^{-1}(V \cap f(A)) \subset f^{-1}(V) \subset f^{-1}(f(A)) \subset f^{-1}(V) \subset f^{-1}(A) \) is countable (since \( f \) is bijective), where \( f^{-1}(y) \in f^{-1}(V) \in \tau \) (since \( f \) is continuous). Hence \( x \notin \text{cl}_{\omega}(A) \), so \( y \notin f(\text{cl}_{\omega}(A)) \) (since \( f \) is bijective).

**Note:** if \( f \) is continuous, it is clear that, \( (\text{cl}(A)) \subset \text{cl}(f(A)) \).
3.17. Lemma

Let \((X, \tau_1, \tau_2), (Y, \sigma_1, \sigma_2)\) be two bitopological spaces, \(f: X \to Y\) bijective, i-open and j-continuous. If \(A\) is an \(ij-\omega p\) open subset of \(X\), then \(f(A)\) is \(ij-\omega p\) open subset of \(Y\).

**Proof:** \(A \subseteq i-\text{int}_\omega(j-\text{cl}A)\) implies \(f(A) \subseteq f(i-\text{int}_\omega(j-\text{cl}A))\), by 3.15 \(f(i-\text{int}_\omega(j-\text{cl}A)) \subseteq \text{int}_\omega(f(j-\text{cl}A))\), by 3.16 \(f(j-\text{cl}A) \subseteq j-\text{cl}(f(A))\), hence, \(f(A) \subseteq f(i-\text{int}_\omega(j-\text{cl}A)) \subseteq \text{int}_\omega(j-\text{cl}(f(A))\). Therefore \(f(A)\) is \(ij-\omega p\) open subset of \(Y\).

**Note:** \(ij-\omega p\) in the above lemma can be replaced by \(ij-\omega h\) \((ij-\omega h)\), \(h= p, s, \alpha, \beta, b\), and the proof will stay similar.

3.18. Theorem

Let \((X, \tau_1, \tau_2), (Y, \sigma_1, \sigma_2)\) be two bitopological spaces, \(f: X \to Y\) bijective, i-open and j-continuous.

If \((Y, \sigma_1, \sigma_2)\) is \(ij-\omega h\) \((ij-\omega h)\) connected then \((X, \tau_1, \tau_2)\) is \(ij-\omega h\) \((ij-\omega h)\) connected too. \((h= p, s, \alpha, \beta, b)\)

**Proof:** One case will be proved, other cases are similar.

Assume that \((Y, \sigma_1, \sigma_2)\) is \(ij-\omega p\) connected, \(f\) is bijective, i-open and j-continuous, and assume that \((X, \tau_1, \tau_2)\) is \(ij-\omega p\) disconnected. Let \(A\) and \(B\) be two non empty disjoint \(ij-\omega p\) open subsets of \(X\) such that \(X=A \cup B\). By 3.17 \(f(A)\) and \(f(B)\) are \(ij-\omega p\) open subsets of \(Y\), they are nonempty and disjoint (since \(f\) is bijective) and \(Y= f(A) \cup f(B)\), which contradicts the assumption that \((Y, \sigma_1, \sigma_2)\) is \(ij-\omega p\) connected. Therefore \((X, \tau_1, \tau_2)\) is \(ij-\omega p\) connected.

References