Separation Axioms Via Turing point of an Ideal in Topological Space

Luay A. Al-Swidi
Mathematics Department, College of Education For Pure sciences
University of Babylon
E-meal:drluayha11@yahoo.com

Ahmed B. AL-Nafee
Mathematics Department, College of Education For Pure sciences
University of Babylon
E-meal:Ahm_math_88@yahoo.com

ABSTRACT

In this paper, we use the concept of the turing point and immersed it with separation axioms and investigate the relationship between them.

Keywords: Turing point, separation axioms.

1. **Introduction and Preliminaries.**

In 2012,Luay A. AL-Swidi and Dheargham A. Al-Sada.[2] introduced and studied the notion of turing point. They defined it as: Let I be an ideal on a topological space (X,T) and $x\in X$. we say that x is a "turing point " of I if $N^c\in I$ for each $N\in N_x$. In the same year, the present author [5] introduced relation between the separation axioms $R_{i\ i=0,1,2and\ 3}$, $T_{i\ ,i=1,2,3\ and\ 4}$, and kernel set in topological space.

Within this paper, the separation axioms $R_{i\ ,i=0,1,}$ and $T_{i\ ,i=0,1,}$ are characterized using a turing point. Further, the axioms $T_{i\ ,i=0,1,2,3\ and\ 4}$. are characterized using a turing point, associated with the axioms R_{i} , i $=_{0,1,2and\ 3}$.

Throughout this paper, spaces means topological spaces on which no separation axioms are assumed unless otherwise mentioned. We define an ideal on a topological space (X,T) at point x by $I_x = \{U\subseteq X : x\in U^c\}$,where U is non-empty set. Let A be asubset of a space X. The closure and the interior of A are denoted by cl(A) and int(A), respectively.

Definition 1.1.

A topological space(X,T) is called

1) R_o –space[1,3,7]if and only if for each open set G and $x \in G$ implies $cl(\{x\}) \subseteq G$.
2) R_1 –space[1,3,7]if and only if for each two distinct point x, y of X with $cl(\{x\})\neq cl(\{y\})$, then there exist
disjoint open sets U,V such that $cl(\{x\}) \subseteq U$ and $cl(\{y\}) \subseteq V$
3) R_2-space [3]if it is property regular space.
4) R_3-space [4]if and only if (X,T) is a normal and R_1-space.
5) Door space[6,5] if every subset of X is either open or closed.
6) Symmetric space [7]if for each two point x, y of X ,$y\in\psi$-cl{x} iff $x\in\psi$-cl{y}.

Remark 1.2[3]

Each separation axiom is defined as the conjunction of two weaker axiom :T_k- space =R_{k-1}-space and T_{k-1} space =R_{k-1}-space and T_0-space , k=1,2,3,4

Remark 1.3[3]

Every R_i –space is an R_{i-1}-space I = 0,1,2,3.

Theorem 1.4[7]

A topological space (X,T) is ψ- R_o –space if and only if for any two points x, y of X .(y$\in$ ψ-cl{x} iff x$\in\psi$-cl{y}).

Theorem 1.5[6]

Every compact Hausdorff space is a T_3-space (and consequently regular).

Theorem 1.6[6]

Every compact Hausdorff space is a normal space (T_3-space).

2.T_i and R_i -Spaces,i =0,1

Lemma 2.1

Let (X,T) be a topological space , for any pair of distinct points x and y of X. Then {y} is closed set if and only if x is not turing point of I_y.

Proof.

Let x,y$\in$ X such that x$\neq$y. Assume that {y} is closed set in X ,so that {y}=cl({y}). But x$\neq$y , we get that x$\notin$cl({y}). Therefore ,there exists an open set U such that, x $\in$ U, U$\cap${y}=$\emptyset$ .So that x $\in$ U, $U^c\notin I_y$,because if $U^c\in I_y$, then y$\in${x}^{c} =U ,that mean U$\cap${y}$\neq\emptyset$,this a contraction. Hence x is not turing point of I_y.

Conversely

Let x,y$\in$ X such that x$\neq$y . Since x is not turing point of I_y then, there exists an open set U such that, x $\in$ U, $U^c\notin I_y$, so y$\notin$U .Thus x $\in$ U, U$\cap$ {y}=$\emptyset$ implies x$\notin$cl({y}).Hence {y} =cl({y}) .Thus {y} is closed set in X .

Theorem 2.2

Let (X,T) be a topological space ,then the following properties are equivalent:

a) (X,T) is a T_0 – space
b) for any pair of distinct points x and y of X, x$\notin$ Tur{y} or y$\notin$ Tur{x}.
c) for any pair of distinct points x and y of X, cl({x})$\neq$cl({y}).

proof

a$\Rightarrow$b. Let x, y $\in$ X with x $\neq$ y . Assume that x$\notin$ Tur{x} or y$\notin$ Tur{x}. By assumption, there exists an open set U such that, x $\in$ U, $U^c\notin I_y$,so y$\notin$U or there exist an open set V such that, y $\in$ V, $V^c\notin I_x$,so x$\notin$V and we have , x $\in$ U, y$\notin$U or y $\in$ V, x$\notin$V .Thus (X,T) is T_0 – space .

b$\Rightarrow$a. Let x,y$\in$ X such that x$\neq$y and (X,T) is T_0 – space, then there exists an open set U such that, x $\in$ U y $\notin$U or there exist an open set V such that y $\in$ V, x$\notin$V and so x $\in$ U, $U^c\notin I_y$ or y $\in$ V, $V^c\notin I_x$ .Thus x is not turing point of I_y or y is not turing point of I_x.Thus x$\notin$ Tur{y} or y$\notin$ Tur{x}.

b$\Rightarrow$c. Let any pair of distinct points x and y of X, x$\notin$ Tur{y} or y$\notin$ Tur{x}. Then by lemma 2.1,then cl({x})={x} or cl({y})={y}.That means x$\in$ cl({x}) and y$\notin$ cl({x}) or y$\in$ cl({x}) and x$\notin$cl({y}). Thus we have x$\in$cl({x}) but x$\notin$ cl({y}).Hence cl({x})$\neq$cl({y}).

c$\Rightarrow$a. Let any pair of distinct points x and y of X, cl({x})$\neq$cl({y}).Then there exists at least one point z $\in$ X such that z $\in$ cl({x}) but z $\notin$ cl({y}). We claim that x $\notin$cl({y}). If x $\in$cl({y}) then {x} $\subseteq$ cl({y}) implies cl({x}) $\subseteq$ Cl({y}). So, z $\in$ cl({y}), which is a contradiction. Hence, x $\notin$ cl({y}). Now, x $\notin$ cl({y}) implies x $\in$ X – cl({y}), which is an open set in X containing x but not y. Hence X is a T_o space.

Theorem 2.3

If a topological space (X,T) is a R_o- space. Then for any pair of distinct points x and y of X, y is not turing point of I_x . .

Proof.

Assume that (X,T) is an R_{o}- space, and let x,y∈ X such that x≠y . By assumption ,then cl({x}) ⊆ V , for each open set V containing x, implies x∉(cl({x}))^c and so cl({x}) ∉I_x ,hence by definition of turing point we get that y is not turing point of I_x.

Remark 2.4

The converse of theorem 2.6,need not be true as seen from the following example .

Example:2.5

Let X ={x,y} and T={∅,{y},X} then y is not turing point of I_x={∅,{y}} ,but X is not a R_{o}- space,.

Remark 2.6

A topological space(X,T) is a symmetric if and only if it is an R_0 space.

Proof

By definition, theorem 1.4 [directly] .

Theorem 2.7

Let (X,T) be a topological space ,then the following properties are equivalent:

a) (X,T) is a T_1 – space
b) for each x ∈ X .Then {x}=Tur{x}
c) for any pair of distinct points x and y of X, x∉ Tur{y} and y∉ Tur{x}.
d) for any pair of distinct points x and y of X, {x}and {y} are closed sets in X
e) for any pair of distinct points x and y of X, Tur{x} ∩Tur{y} =∅
f) for any pair of distinct points x and y of X , (X,T) is symmetric space, with x∉ Tur{y} or y∉ Tur{x}.

Proof.

a⇒b. Assume that (X,T) is a T_1 – space, let x ∈X .We will prove that {x} = Tur{x}. It is clear that {x} ⊆Tur{x} .Let y ∈X be such that x≠y. By assumption, there exist an open set V such that, y ∈V and x∉V so V^c∉ I_x , that is , y is not turing point of I_x . and so ,y ∉Tur{x} . Thus Tur{x} ⊆ {x} . Therefore {x} = Tur{x}.

b⇒a. Assume that {x} = Tur{x}, for each x ∈ X. Let (X,T) be not T_1 – space and let x,y∈ X such that x≠y .Then for each open set V containing x, it contains y,so we get that y is not turing point of I_x, thus ,Tur{x} contains y ,it is follows that, Tur{x}≠{x} ,this is contraction with assumption.Thus (X,T) is a T_1 – space.

b⇒c. Let x,y∈ X such that x≠y . since {x} = Tur{x}, for each x ∈ X. So that x∉ Tur{y} and y∉ Tur{x}.

c⇒d. Let any pair of distinct points x and y of X, x∉ Tur{y} and y∉ Tur{x}. Then by lemma 2.1,then {x} and{y} are closed subsets of X.

d⇒e. Let any pair of distinct points x and y of X. By assumption , then {x} and {y} are closed sets and as such $\{x\}^c$ and $\{y\}^c$ are open sets. Thus y ∈ $\{x\}^c$ but x ∉ $\{x\}^c$ and x ∈ $\{y\}^c$ but y ∉ $\{y\}^c$. Therefore x∈ Tur{x} but y ∈Tur{y}.Thus Tur{x} ∩Tur{y} =∅

e⇒a. Let x≠y , for each x,y∈X and Tur{x} ∩Tur{y} =∅ , such that (X,T) is not T_1 – space. Then for each open set V containing x, it contains y, so x, y ∈V ,implies V^c∈I_x and V^c∈ I_y ,so we get that x∈ Tur{x} and x ∈Tur{y}.So that Tur{x} ∩Tur{y} ≠∅ .This is a contraction . Hence (X,T) is a T_1 – space .

a⇒f. Assume that (X,T) is a T_1 – space. Then by remark 2.6, it is a R_{o}- space and T_0 –space . Hence (X,T) is symmetric space and y∉ Tur{x} and x ∉Tur{y} [Theorem 2.2 and remark 2.6].

f⇒a. Let $x \neq y$, for each $x,y \in X$ and (X,T) is symmetric space, with $x \notin$ Tur{y} or $y \notin$ Tur{x} .Then by remark 2.6, it is a R_{o}- space . Also it is a T_{o}–space [since $x \notin$ Tur{y} or $y \notin$ Tur{x}] . Thus it is a T_1–space[By remark 1.2].

Corollary 2.8

A topological space(X,T) is a T_1 – space if and only if ,for each $x \in X$. Then cl({x}) = Tur{x}.

Proof

By theorem 2.7 [directly] .

Theorem 2.9

A topological space(X,T) is R_1 – space if and only if ,for each $x,y \in X$ such that $x \neq y$ with cl({x})≠cl({y}),then ,there exist disjoint open sets U,V such that cl(Tur{x})⊆U and cl(Tur{y})⊆V.

Proof .

Assume that $x \neq y$ with cl([x})≠cl({y}) and(X,T)is an R_1–space for each $x,y \in X$. By assumption,then there exist disjoint open sets U,V such that cl({x}) ⊆U and cl({y}) ⊆ V . Also (X,T) is a T_1 – space[Remark 1.2] and So by corollary2.11,then cl({x})=Tur{x} and cl({y})=Tur{x}implies cl(Tur{x})=cl(cl({x}))=cl({x})⊆U,and=cl(Tur{y})=cl(cl({y}))=cl({y})⊆V.

Conversely

Assume that $x \neq y$ with cl({x})≠cl({y}) ,for each $x,y \in X$, then there exist disjoint open sets U,V such that cl(Tur{x})⊆U and cl(Tur{y})⊆V.Since,{x}⊆Tur{x}and{y}⊆Tur{y}.Then,cl({x})⊆cl(Tur{x})= cl(Tur{x})⊆ U andcl({y})⊆cl(Tur{y} ⊆ V. Therefore (X,T) is R_1 – space.

3.T_i ,i=1,2,3 and 4. associated with the axioms R_i , i =0,1,2and 3

Theorem 3.1

For an R_1- space (X,T) the following properties are equivalent:

a) (X,T) is a T_2- space
b) for each $x \in X$, {x} = Tur{x}
c) for $x, y \in X$ with $x \neq y$, Tur{x}∩Tur{y}=∅
d) for $x,y \in X$ with $x \neq y$, $x \notin$Tur{y} and $y \notin$Tur{x}.

Proof .

a⇒b. Let $x \in X$ .Since (X,T) is a T_2-space ,then it is a T_1-space. Therefore ,by theorem 2.7, part 'b' , {x} =Tur{x} for each $x \in X$.

b⇒a. Assume that {x}=Tur{x}, for each $x \in X$. Then (X,T)is a T_1- space[Theorem 2.7, part 'b'].But (X,T) is an R_1-space, then (X,T) is a T_2-space[Remark 1.2].

a⇒c. Since (X,T) be a T_2-space,then it is aT_1-space,and so Tur{x} ∩Tur{y}=∅ [Theorem 2.7, part'e'], for any pair of distinct points x and y of X.

c⇒a. Let x and y be two distinct points in X. By assumption,(X,T) is a T_1-space [Theorem 2.7 part 'e']. But (X,T) is an R_1-space, implies (X,T) is a T_2-space [Remark 1.2].

c⇒d. Let x and y be two distinct points in X. By assumption ,(X,T) is a T_{1-}space[Theorem 2.7 part 'e'].Therefore by theorem2.7 part 'c', then x∉Tur{y} and y∉Tur{x}.

d⇒a. Let x and y be two distinct points in X .By assumption, then (X,T) is a T_{1-}space [By theorem2.7 part'c'].But (X,T) is an R_1-space .Hence (X,T) is a T_{2-}space [Remark 1.2].

Remark 3.2

Observe that every T_2 –space is an T_1-space and every T_1 –space is an T_0-space the converse, need not be true. But it is true generally, if (X,T) is an R_1-space as seen from the following theorem.

Theorem 3.3

For an R_{1-} space (X,T) the following properties are equivalent:

a) (X,T) is a T_{2-} space
b) for each x,y∈ X such that x≠y, either x∉Tur{y} or y∉Tur{x}.

proof.

a ⇒b. Let x and y be two distinct points in X. Since (X,T) be a T_{2-} space ,then (X,T) be a T_{0-} space by theorem2.2 part 'b' ,either x∉Tur{y}or y∉Tur{x}.

b ⇒a.

Let x and y be two distinct points in X, where x∉ Tur{y} or y∉ Tur{x}, this impels that (X,T) is a T_{0-}space[By theorem2.2].But (X,T) is an R_1-space .Hence (X,T) is a T_{2-}space. [Remark 1.2].

Remark 3.4

Observe that every T_3 –space is an T_2-space ,the converse, need not be true. But it is true generally, if (X,T) is an R_2-space as seen from the following corollary.

Corollary 3.5

For an R_{2-} space (X,T) the following properties are equivalent:

a) (X,T) is a T_{3-} space
b) for x ∈X ,{x}=Tur{x}.
c) for x, y ∈ X with x ≠ y, Tur{x}∩Tur{y}=∅
d) or x,y ∈X with x≠y , x∉Tur{y} and y∉Tur{x}.

proof

By remark 1.3 , remark1.2 and theorem3.2.

Theorem 3.6

For an R_{2-} space (X,T) the following properties are equivalent:

(a) (X,T) is a T_{3-} space
(b) for each x,y∈ X such that x≠y, either x∉Tur{y}or y∉Tur{x}.

proof

a ⇒b. Let x and y be two distinct points in X. Since (X,T) be a T_{3-} space ,then (X,T) be a T_{0-} space by theorem2.2 part 'b' ,either x∉Tur{y}or y∉Tur{x}.

b ⇒a. Let x and y be two distinct points in X x∉ Tur{y} or y∉ Tur{x}, this impels that (X,T) is a T_0-space[By theorem2.2].But (X,T) is an R_2-space .Hence (X,T) is a T_3-space. [Remark 1.2].

Theorem 3.7

For a compact R_1-space(X,T) the following properties are equivalent:

a) (X,T) is a T_{3-} space
b) for x,y ∈X with x≠y , x∉Tur{y} and y∉Tur{x}.
c) for x, y ∈ X with x ≠ y, Tur{x}∩Tur{y}=∅
d) for x ∈X ,{x}=Tur{x}.

proof

a⇒b. Let x and y be two distinct points in X. By assumption ,then (X,T) is a T_{1-} space , and so, for each x,y ∈ X such that x≠y, x∉Tur{y}and y∉Tur{x}[Theorem2.7 part 'c'].

b⇒a. Let x and y be two distinct points in X .By assumption, then (X,T) is a T_1-space[By theorem2.7part 'c']. But (X,T) is R_1-space. Hence (X,T) is T_{2-} space [Remark 1.2].Which implies that (X,T) is a compact T_{2-} space. So that (X,T) is a T_{3-} space [Theorem 1.5].

b⇒c. Let x,y∈ X such that x≠y, x∉Tur{y}and y∉Tur{x},and so by theorem2.7part 'c', then (X,T) is a T_1-space,and so for each x,y∈ X such that x≠y, Tur{x}∩Tur{y}=∅[Theorem2.7 part 'e'].

c⇒d. Let x ∈X. By assumption, then (X,T) is a T_{1-} space [by theorem2.7 part 'e'], and so for each x ∈ X ,then {x} = Tur{x} [theorem2.7 part 'b'].

d⇒a. Let x,y∈ X such that x≠y. By assumption, then (X,T) is a T_{1-} space [By theorem2.7 part 'b']. But (X,T) is a compact R_1-space and so (X,T) is a compact T_{2-} space [Remark 1.2] . Hence (X,T) is a T_{3-} space [by theorem 1.5].

Remark 3.8

Observe that every T_4 –space is an T_3-space,the converse, need not be true. But it is true generally, if (X,T) is a compact R_1-space as seen from the following corollary.

Corollary 3.9

For a compact R_{1-} space (X,T) the following properties are equivalent:

a) (X,T) is a T_{4-} space
b) for x,y ∈X with x≠y , x∉Tur{y} and y∉Tur{x}.
c) for x, y ∈ X with x ≠ y, Tur{x}∩Tur{y}=∅
d) for x ∈X ,{x}=Tur{x}.

proof

ISSN 1661-464X

By Remark 1.2,theorem2.7 part 'b', theorem 3.7 part 'b'and theorem1.6 [directly].

Theorem 2.10

For a compact R_{1-} space (X,T) the following properties are equivalent:

(a) (X,T) is a T_{3-} space

(b) for x,y ∈X with x≠y ,either x∉Tur{y} or y∉Tur{x}.

proof.

a ⇒b. Let x and y be two distinct points in X. Since (X,T) be a T_{3-} space ,then (X,T) be a T_{0-} space, hence by theorem2.2 part 'b' ,either x∉Tur{y}or y∉Tur{x}.

b ⇒a. Let x and y be two distinct points in X where x∉ Tur{y} or y∉ Tur{x}, hence by theorem 2.2, (X,T) is a T_0-space. In addition (X,T) is a compact R_1-space. Hence, by remark 1.2, (X,T) is a compact T_2-space. Hence, by theorem1.5, (X,T) is a T_3-space.

Corollary 3.11

For a compact R_{1-} space (X,T) the following properties are equivalent:

(a) (X,T) is a T_{4-} space

(b) for x,y ∈X with x≠y ,either x∉Tur{y} or y∉Tur{x}.

proof

By theorem 3.8 and remake 3.6 [directly].

REFERENCE

[1] N. A. Shanin, On separation in topological spaces, Dokl. Akad. Nauk SSSR, 38 (1943), 110–113.

[2] L. A. AL-Swidi and Dheargham A. Al-Sada "Turing Point of Proper Ideal" Archive Des Science (Impact factor 0.474) ISSN:1661-464X Volume 65 No. 7 Jul(2012), 219-220.JST Journal. All rights reserved.

[3] A.S. Davis, "Indexed Systems of Neighborhoods for General Topological Spaces," The Amer. Math. Monthly, 68,886 – 893 (1961).

[4] L. A. AL-Swidi and Mohammed, B.(2012)."Separation axioms via kernel set in topological spaces" Archive Des sciences , Vol .65, No.7 ,pp 41 -48.

[5] T. Noiri, A. Al-Omari and M. S. M. Noorani " ,Weak forms of open sets and decomposition of continuity", E.J.P.A.M. 2(1): 73-84 (2009).

[6] J.N.Sharma.(1977)."general topology" Krishna prakashan, Meerut(U.P).

[7] Bishwambhar Roy and M.N.Mukherjee. A unified theory for $R_{0,}$ R_1 and certain other separation properties and their variant forms ,Bol .Soc.paran. Mat.(3s)v.28.2(2010):15-24