Are The Orlicz Spaces Generated By Dilatory Function And Their Duals Are Banach Spaces

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ABSTRACT: The Orlicz spaces generated by dilatory functions are only quasi-Banach spaces contrast to those generated by Orlicz functions which are Banach spaces, and their duals are Banach space also .

1-Introduction :

We shall introduce a background of the Orlicz space the word Orlicz came from the name of the mathematician Wiadyslaw Roman Orlicz.

Orlicz spaces are generalization of L_p space their definition are very well known :if $(\Omega,\mathcal{F},\mu)$ is a measure space ,and $1 \leq P \leq \infty$ then for any measurable function $f : \Omega \to \mathbb{C}$ the L_p-norm is defined to be

$$\| f \|_p = \left(\int_\Omega |f(w)|^p \, d\mu(w) \right)^{\frac{1}{p}} \qquad \text{for } p < \infty \ .$$

And $\|f\|_\infty = \text{ess sup}_{w\in\Omega} |f(w)|$ for $p = \infty$

Then we define the Banach space $L_p(\Omega,\mathcal{F},\mu)$ to be the vector space of all measurable function $f: \Omega \longrightarrow \mathbb{C}$ for which $\|f\|_p$ is finite.

Now. If $F: [0,\infty)\to[0,\infty)$ is an Orlicz function where F is non-decreasing convex with $F(0)=0$ then we define the Luxemburg norm by $\|f\|_F = \inf\left\{c: \int_\Omega F(\frac{|f(w)|}{c})d\mu \leq 1\right\}$ for all measurable function f and define Orlicz space $L_F(\Omega,\mathcal{F},\mu)$ to be those measurable function f for which $\|f\|_F$ is finite the Orlicz space L_F is a true generalization of L_p at least for $p < \infty$. If $F(t) = t^p$, then $L_F = L_p$ with quality norms.

We shall not work with this definition of the Orlicz space , however , but with different equivalent definition . this definition we give in the following section .

2 – Definitions : We first define Φ- function . these replace the notion of Orlisz – function in our discussions .

Definition (2-1) [Montgomery , 1999] : A Φ – function is a function

$F : [0 , \infty) \to [0 , \infty)$ such that

i) $F(0) = 0$

ii) $\lim_{n\to\infty} F_{(t)} = \infty$

iii) F is strictly increasing

iv) F is continuous

However we will often desire that the function F has some control on its growth both from above and below for this reason we will often require that F be dilatory .

We will say that a Φ – function F is dilatory if for some $K_1, K_2 > 1$ we have $F(K_1 t) \geq K_2 F(t)$ for all $0 \leq t < \infty$

We will say that F satisfies the Δ_2-conditon if F^{-1} is dilatory

The definition of Φ-function is slightly more restrictive than that of an Orlicz function in that we insist that F be strictly increasing the notion of dilatory replaces the notion of convexity

Definition(2-2) [Montgomery] : if $(\Omega, \mathcal{F}, \mu)$ is a measure space and F is

Φ – function , then we define Luxemburg functional of a measurable function f by

$$\|f\|_F = \inf\left\{c: \int_\Omega F\left(\frac{|f(w)|}{c}\right) d\mu(w) \leq 1\right\}$$

for every measurable function f ,we define the Orlicz space L_F to be the vector space of measurable function f ,for which $\|f\|_F < \infty$ modulo functions that are zero almost everywhere .

Definition(2-3) [Cong and Yongjin , 2008] : quasi – norm on a(real or complex) vector space X is anon-negative real –valued function on X satisfying :

(i) $\|x\| = 0$ if and only if $x = 0$

(ii) $\|\lambda x\| = |\lambda| \, \|x\|$ for all $x \in X$ and $\lambda \in R$

(iii) $\|x + y\| \leq K [\|x\| + \|y\|]$ for some fixed $K \geq 1$ and all $x, y \in X$

3- Results :

Theorem (3-1) : If F(t) is Φ– function satisfy dilatory condition , then L_F is a quasi – Banach space .

Proof : -

(i) Let $\|x\|_F = 0$, since $c > 0$ *and* $\|x\|_F = 0$,

then c is very small and greater than zero ,so x must equal to zero.

If x =0 and $c > 0$ so $\|x\|_F$ must be zero.

(ii) since F is dilatory ,then $F(K_1 w) \geq K_2 F(w)$ and since F is increasing so K_1=K_2

hence $\int_\Omega F(\frac{|K_1 f(w)|}{c}) d\mu(w) \geq K_1 \int_\Omega F(\frac{|f(w)|}{c}) d\mu(w)$

$$\inf\left\{c: \int_{\Omega} F(\frac{K_1|f(w)|}{c})d\mu(w)\right\} \leq K_1 \inf\left\{c: \int_{\Omega} F(\frac{|f(w)|}{c})d\mu(w)\right\} .$$

Since $\int_{\Omega} F(\frac{K_1|f(w)|}{c})d\mu \leq 1$

So $K_1 \int_{\Omega} F(\frac{|f(w)|}{c})d\mu \leq 1$.

Hence $\|K_1 f\|_F = \inf \{ c : \int_{\Omega} F(\frac{|K_1 f(w)|}{c})d\mu(w) \leq 1 \}$

$= |K_1| \inf \{ c : \int_{\Omega} F(\frac{|f(w)|}{c})d\mu(w) \leq 1 \}$

$= |K_1| \; \|f\|_F$.

iii) Since F is dilatory we have $K_2 \int_{\Omega} F(\frac{|x+y|}{\|x\|+\|y\|})d\mu \leq \int_{\Omega} F(\frac{K_1|x+y|}{\|x\|+\|y\|})d\mu$

Since $x + y > x$ *then* $K_1(x + y) > K_1(x)$

and $F(K_1(x + y)) > F(K_1(x))$

Since $\|x\| + \|y\| > \|x\|$

So $\frac{1}{\|x\|+\|y\|} < \frac{1}{\|x\|}$

Then $\int_{\Omega} F(\frac{|x+y|}{\|x\|+\|y\|})d\mu \leq \frac{1}{K_2}[(\int_{\Omega} F(\frac{K_1|x+y|}{\|x\|+\|y\|})d\mu]$

$\leq \frac{K_1}{K_2}[\int_{\Omega} F(\frac{|x|}{\|x\|})d\mu + \int_{\Omega} F(\frac{|y|}{\|y\|})d\mu]$.

Since $\int_{\Omega} F(\frac{|x|}{\|x\|})d\mu \leq 1$ and $\int_{\Omega} F(\frac{|y|}{\|y\|})d\mu \leq 1$.

So $\inf \{ (\|x\| + \|y\|) : \int_{\Omega} F(\frac{|x+y|}{\|x\|+\|y\|})d\mu \leq 1\}$.

Hence $\|x + y\| \leq \frac{K_1}{K_2} [\|x\| + \|y\|]$

Let $K = \frac{K_1}{K_2}$ So $\|x + y\| \leq K[\|x\| + \|y\|]$.

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