On paracompact in bitopological spaces.

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SUMMARY. We modify the concept of paracompactness for spaces with two topologies and obtain several results concerning paracompact in bitopological spaces.

1-Introduction

Bitopological space , initiated by Kelly [ 7 ], is by definition a set equipped with two non identical topologies , and it is denoted by (X, τ,μ) where τ and μ are two topologies defined on X .

A sub set F of a topological space (X , τ) is Fσ [ 11 ] if it is a countable union of τ-closed set . We will denote to such set by τ- Fσ.

Let (X , τ) be a topological space . A cover (or covering) [ 3 ] of a space X is a collection $U = \{U_\lambda : \lambda \in \Delta\}$ of subset of X whose union is the whole X .

A sub cover of a cover U [ 3 ] is a sub collection v of u which is a cover .

An open cover of X [ 3 ] is a cover consisting of open sets , and other adjectives apply to subsets of X apply similarly to covers .

For an infinite cardinal number m , if the collection $U = \{U_\lambda : \lambda \in \Delta\}$ consists of at most m sub-sets, we say that it has cardinality $\leq m$ or simply card. $\leq m$ . Some times this collection is denoted by $|U| \leq m$(or)$|\Delta| \leq m$ .

If a sub set A of X is consisting of at most m elements we say that A has cardinality $\leq m$ (or with cardinality $\leq m$ ) , and is denoted by $|A| \leq m$ . A bitopological space (X, τ, μ) is called (m) (τ-μ) compact if for every τ-open cover of X, (with cardinality $\leq m$ ), it has μ-open sub-covers . The function $f : (X, \tau, \mu, \rho) \rightarrow (Y, \tau', \mu', \rho')$ is said to be $(\tau-\tau')$-close$[(\tau-\tau')$continuous] function if the image [inverse image of each τ-closed[τ-open] is τ'-closed [τ-open in X] in Y .

Let $U=\{U_\lambda : \lambda \in \Delta\}$ and $V=\{V_\gamma : \gamma \in \Gamma\}$ be two coverings of X , V is said to refine (or to be a refinement of ) U , if for each $V_\gamma$ there exists some $U_\lambda$ with $V_\gamma \subseteq U_\lambda$ .

If $W = \{W_\delta : \delta \in \Omega \}$ refine two covers $U, V$ of $X$, then it is called common refinement [2]. A family $U = \{U_\lambda : \lambda \in \Delta \}$ of sets in a space $(X, \tau)$ is called locally finite, if each point of $X$ has a neighborhood $V$ such that $V \cap U_\lambda \neq \phi$ for at most finitely many indices $\lambda$. In other word $V \cap U_\lambda = \phi$ for all but a finite number of $\lambda$. A family $U$ of set in a space $(X, \tau)$ is called $\sigma$-locally finite if

$$U = \bigcup_{n=1}^\infty U_n$$

where each $U_n$ is a locally finite collection in $X$.

A bitopological space $(X, \tau, \mu)$ is called pairwise Hausdorff if for every two distinct points $x$ and $y$ of $X$, there exist $\tau$-open set $U$ and a $\mu$-open set $V$ such that $x \in U, y \in V$ and $U \cap V = \phi$.

A bitopological space $(X, \tau, \mu)$ is called (m)-($\tau, \mu, \mu$)-regular if for every point $x$ in $X$ and every $\tau$-closed set $A$ with $|A| \leq m$ such that for $x \in A$, there exist two $\mu$-open sets $U, V$ such that $x \in U, A \subseteq V$, and $U \cap V \neq \phi$.

Clearly every ($\tau, \mu, \mu$)-regular space is $m(\tau, \mu, \mu)$-regular space.

A bitopological space $(X, \tau, \mu)$ is called (m)-($\tau, \mu, \mu$)-normal if for every pair disjoint $\tau$-closed sets $A, B$ of $X$, with $|A| \leq m, |B| \leq m$ there exist two $\mu$-open sets $U, V$ such that $A \subseteq U, B \subseteq V$, and $U \cap V = \phi$.

Clearly every ($\tau, \mu, \mu$)-normal space is $m(\tau, \mu, \mu)$-normal.

A topological space $(X, \tau)$ is said to be :

1- m-paracompact [9], if every open cover of $X$ with card. $\leq m$ has a locally finite open refinement.

2- paracompact[4], if every open cover of $X$ has a locally finite open refinement.

3- (m-) semiparacompact, if every open cover of $X$ (with card. $\leq m$) has a $\sigma$-locally finite open refinement.

4- (m-) a-paracompact[1] if every open cover of $X$ with card. $\leq m$ has a $\alpha$-locally finite refinement not necessary either open or closed.

2-Main Results

2.1-Definition

A bitopological space $(X, \tau, \mu)$ is called (m-) ($\tau - \mu$) paracompact w.r.t $\mu$, if for every $\tau$-open cover $U = \{U_\lambda : \lambda \in \Delta \}$ of $X$ (with card. $\leq m$) has a $\mu$-open refinement $V = \{V_\gamma : \gamma \in \Gamma \}$ which is locally finite w.r.t $\mu$. 

2.2 - Proposition

Every (τ -µ) paracompact w.r.t µ bitopological space (X, τ, µ) is m (τ -µ) paracompact w.r.t µ.

2.3 - Definition

A bitopological space (X, τ, µ) is called (m-) (τ -µ) semiparacompact w.r.t µ, if every τ-open cover \( U = \{ U_\lambda : \lambda \in \Delta \} \) of X (with card. \( \leq m \)) has a µ-open refinement \( V = \{ V_\gamma : \gamma \in \Gamma \} \) which is σ-locally finite w.r.t µ.

2.4 - Proposition

Every (τ -µ) semiparacompact w.r.t µ bitopological space (X, τ, µ) is m(τ -µ) semiparacompact w.r.t µ.

2.5 - Theorem

Every m(τ-µ) paracompact w.r.t µ bitopological space (X, τ, µ) is m(τ -µ) semiparacompact w.r.t µ.

2.6 - Corollary

Every (τ-µ) paracompact w.r.t µ bitopological space (X, τ, µ) is (τ-µ) semiparacompact w.r.t µ.

2.7 - Corollary

Every (τ-µ) paracompact w.r.t µ bitopological space (X, τ, µ) is m(τ -µ) semiparacompact w.r.t µ.

2.8 - Definition

A bitopological space (X,τ, µ) is called (m-) (τ -µ) -a-paracompact w.r.t µ, if for every τ-open cover \( U = \{ U_\lambda : \lambda \in \Delta \} \) of X (with card. \( \leq m \)) has a refinement \( V = \{ V_\gamma : \gamma \in \Gamma \} \) of U not necessarily either µ-open or µ-closed which is locally finite w.r.t µ.

2.9 - Proposition

Every (τ-µ) -a-paracompact w.r.t µ bitopological space (X, τ, µ) is m (τ -µ) -a-paracompact w.r.t µ.

2.10 - Theorem

Every m(τ -µ) semiparacompact w.r.t µ bitopological space (X, τ, µ) is m (τ -µ) -a-paracompact w.r.t µ.

Proof

Suppose that $(X, \tau, \mu)$ be $m(\tau\mu)$semiparacompact w.r.t. $\mu$ space. Let $U = \{U_\lambda : \lambda \in \Delta\}$ be a $\tau$-open cover of $X$ with $\text{card} \leq m$, then $U$ has $\mu$-open refinement $V$ of $U$ which is $\sigma$-locally finite w.r.t. $\mu$, such that

$$V = \bigcup_{n=1}^{\infty} V_n$$

where each $V_n$ is $\mu$-open collection which is locally finite w.r.t. $\mu$, say $V_n = \{V_{n\beta} : \beta \in B\}$. For each $n$, let

$$W_n = \bigcup_{\beta} V_{n\beta}$$

then $W_n$ is $\mu$-open set. Since

$$X = \bigcup_{\beta} \left( \bigcup_{n=1}^{\infty} V_{n\beta} \right) = \bigcup_{n=1}^{\infty} \left( \bigcup_{\beta} V_{n\beta} \right) = \bigcup_{n=1}^{\infty} W_n$$

Then the collection $W = \{W_n | n \in \mathbb{N}\}$ is $\mu$-open cover of $X$.

Define

$$A_i = W_i / \bigcup_{j\neq i} W_j$$

where $i = 1, 2, \ldots$

then $A = \{A_n : n \in \mathbb{N}\}$ is a collection of sets that are not necessarily either $\mu$-open or $\mu$-closed. Then $A$ is cover of $X$, a refinement of $W$ and locally finite w.r.t. $\mu$. Hence $(X, \tau, \mu)$ is $m(\tau\mu)$-a-paracompact w.r.t $\mu$.

In the same way we can prove the following corollaries.

2.11 - Corollary

Every $(\tau\mu)$semiparacompact w.r.t. $\mu$ bitopological space $(X, \tau, \mu)$ is $(\tau\mu)$-a-paracompact w.r.t $\mu$.

2.12 - Corollary

Every $(\tau\mu)$semiparacompact w.r.t. $\mu$ bitopological space $(X, \tau, \mu)$ is $m(\tau\mu)$-a-paracompact w.r.t $\mu$.

2.13 - Corollary

Every $m(\tau\mu)$paracompact w.r.t. $\mu$ bitopological space $(X, \tau, \mu)$ is $m(\tau\mu)$-a-paracompact w.r.t $\mu$.

2.14 - Corollary

Every $(\tau\mu)$paracompact w.r.t. $\mu$ bitopological space $(X, \tau, \mu)$ is $(\tau\mu)$-a-paracompact w.r.t $\mu$.

2.15 - Corollary

Every \((\tau,\mu)\) paracompact w.r.t. \(\mu\) bitopological space \((X,\tau,\mu)\) is \(m(\tau,\mu)\) -a-paracompact w.r.t. \(\mu\).

The following diagram show the relation a among the spaces which have been studied above.

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2.16 - Theorem

Let \((X,\tau,\mu)\) be an \(m(\tau,\mu)\) paracompact w.r.t. \(\mu\) and pairwise Hausdorff space such that every \(\tau\)-closed set in \((X,\tau,\mu)\) has \(\leq m\), then \((X,\tau,\mu)\) is \(m(\tau,\mu,\mu)\)-regular space.

Proof

Suppose that. \((X,\tau,\mu)\) be an \(m(\tau,\mu)\) paracompact w.r.t. \(\mu\) space, \(A\) a \(\tau\)-closed set in \((X,\tau,\mu)\) having \(\leq m\), and \(x \in X / A\).

Since \((X,\tau,\mu)\) is pairwise Hausdorff, then for each \(y \in A\), we can find a \(\tau\)-open set \(V_y\) and a \(\mu\)-open set \(U_y\), such that \(x \in U_y\), and \(U_y \cap V_y = \emptyset\) the collection \(\Pi = \{V_y : y \in A\} \cup \{X/A\}\) form a \(\tau\) – open cover of \(X\) having \(\leq m\).

\(\Pi\) has a \(\mu\)-open refinement \(W = \{W_y : y \in \Gamma\}\) which is locally finite-w.r.t. \(\mu\).

Set

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\[ V = \bigcup_{\gamma \in \Gamma} \{ W_\gamma : W_\gamma \cap A \neq \emptyset \} \]

then \( V \) is \( \mu \)-open set containing \( A \).

Since the \( \mu \)-open cover \( W \) is locally finite. w.r.t. \( \mu \), then \( x \) has a \( \mu \)-neighborhood \( U^* \) which meet only a finite number of \( W_\gamma_1, \ldots, W_\gamma_n \). If some \( W_\gamma_i, \ i=1,2,\ldots,n \) meets \( A \)
i.e. \( W_\gamma_i \cap A \neq \emptyset \), then \( W_\gamma_i \subset X/A \) is impossible thus there exists \( W_\gamma_i \) such that \( W_\gamma_i \subset V_{\gamma_i} \).

Set
\[
U = U * \bigcap_{i=1}^{n} W_\gamma_i
\]

then \( x \in U \) and \( U \) is a \( \mu \)-open set then \( U \cap V = \emptyset \). Therefore the bitopological space \((X, \tau, \mu)\) is \( m(\tau, \mu) \)-regular.

2.17 - Corollary

If \((X, \tau, \mu)\) be a \((\tau, \mu)\) paracompact w.r.t. \( \mu \), and pairwise Hausdorff then \((X, \tau, \mu)\) is \((\tau, \mu, \mu)\)–regular.

2.18 - Theorem

If \((X, \tau, \mu)\) is an \( m(\tau, \mu) \) paracompact w.r.t. \( \mu \), and pairwise Hausdorff space, such that every \( \tau \)-closed set in \((X, \tau, \mu)\) has card. \( \leq m \), then \((X, \tau, \mu)\) is \( m(\tau, \mu, \mu) \)-normal.

\textit{proof}

Suppose that \((X, \tau, \mu)\) be an \( m(\tau, \mu) \) paracompact w.r.t. \( \mu \). Let \( A \) and \( B \) be disjoint \( \tau \)-closed sets in \((X, \tau, \mu)\) such that they have card. \( \leq m \). Since \((X, \tau, \mu)\) is pairwise Hausdorff, then for each \( x \in A, y \in B \) we can find a \( \tau \)-open set \( U_x \) and a \( \mu \)-open set \( V_x \), such that \( x \in U_x, y \in V_x, \) and \( U_x \cap V_x = \emptyset \). Then
\[
\Pi = \{ U_x : x \in A \} \cup \{ X/A \}
\]
form a \( \tau \)-open cover of \( X \) having card. \( \leq m \). Then \( \Pi \) has a \( \mu \)-open refinement \( W = \{ W_\gamma : \gamma \in \Gamma \} \) which is locally finite w.r.t. \( \mu \).

Set
\[
U = \bigcup_{\gamma \in \Gamma} \{ W_\gamma : W_\gamma \cap A \neq \emptyset \}.
\]

Then \( U \) is \( \mu \)-open set contains \( A \).

For each \( y \in \mathcal{B} \), we can find \( \mu \)-open nhd \( H_y \) which meets only a finite number of \( \mathcal{W}_{\gamma_{i(y)}}(\cdot) \), say \( \mathcal{W}_{\gamma_{i(y)}}(\cdot) \) meeting \( A \), i.e \( \mathcal{W}_{\gamma_{i(y)}}(\cdot) \cap A \neq \emptyset \) then \( \mathcal{W}_{\gamma_{i(y)}}(\cdot) \subset X/A \) is impossible. Thus there exists \( U_{x_i} \) such that \( \mathcal{W}_{\gamma_{i(y)}}(\cdot) \subset U_{x_i} \) for \( x_i \in A \).

Set \[ G_y = H_y \cap \left( \bigcap_{i=1}^{n} V_{x_i} \right) \]

then \( G_y \) is a \( \mu \)-open set which contains \( y \) but does not meet \( U \).

Let \[ V = \bigcup_{y \in \mathcal{B}} G_y . \]

Then \( V \) is a \( \mu \)-open set, and \( \mathcal{B} \subset V \) and \( U \cap V = \emptyset \). Therefore \( (X, \tau, \mu) \) is \( m(\tau, \mu, \mu) \)-normal.

2.19 - Corollary

If \( (X, \tau, \mu) \) be a \( (\tau, \mu) \) paracompact w.r.t \( \mu \), and pairwise Hausdorff space then it is \( (\tau, \mu, \mu) \)–normal.

2.20 - Theorem

Let \( (X, \tau, \mu) \) be a bitopological space and let \( (Y, \tau_Y, \mu_Y) \) be a \( \tau \)-closed subspace of \( (X, \tau, \mu) \). If \( (X, \tau, \mu) \) is \( m(\tau, \mu) \) paracompact w.r.t. \( \mu \), then \( (Y, \tau_Y, \mu_Y) \) is \( m(\tau_Y, \mu_Y) \) paracompact w.r.t. \( \mu_Y \).

Proof

Suppose that \( (Y, \tau_Y, \mu_Y) \) be a \( \tau \)-closed subspace of \( m(\tau, \mu) \) paracompact w.r.t. \( \mu \) space \( (X, \tau, \mu) \). Show that \( (Y, \tau_Y, \mu_Y) \) is \( m(\tau_Y, \mu_Y) \) paracompact w.r.t. \( \mu_Y \).

Let \( U = \{ U_{\lambda} : \lambda \in \Delta \} \) be a \( \tau \)-open cover of \( Y \) with card. \( \leq m \).

Since \( U_{\lambda} \) is \( \tau_Y \)-open subset of \( Y \), there is \( \tau \)-open subset \( V_{\lambda} \) of \( X \) such that each \( U_{\lambda} = V_{\lambda} \cap Y \). The collection \( \prod = \{ V_{\lambda} : \lambda \in \Delta \} \bigcup \{ X/Y \} \) form a \( \tau \)-open cover of \( X \) with card. \( \leq m \). By hypothesis \( \prod \) has \( \mu \)-open refinement \( W = \{ W_{\gamma} : \gamma \in \Gamma \} \) which is locally finite w.r.t. \( \mu \).

Now, let \( A = \{ W_{\gamma} \cap Y : \gamma \in \Gamma \} \), then \( A \) is a collection of \( \mu_Y \)-open subset of \( Y \), hence \( A \) is a cover \( Y \) and refine \( U \) locally finite w.r.t. \( \mu \). Therefore \( (X, \tau_Y, \mu_Y) \) is \( m(\tau_Y, \mu_Y) \) paracompact w.r.t. \( \mu_Y \).

2.21 - Corollary
Let \((X, \tau, \mu)\) be a bitopological space and let \((Y, \tau_Y, \mu_Y)\) be a \(\tau\)-closed subspace of \((X, \tau, \mu)\). If \((X, \tau, \mu)\) is \((\tau, \mu)\)-paracompact w.r.t \(\mu\), then \((Y, \tau_Y, \mu_Y)\) is \((\tau_Y, \mu_Y)\)-paracompact w.r.t \(\mu_Y\).

2.22 - Theorem

Let \((X, \tau, \mu)\) be a bitopological space and let \(\mathcal{X} = \{X_i : X_i \in \tau \cap \mu : i \in I\}\) be a partition of \(X\). The space \((X, \tau, \mu)\) is \(m(\tau, \mu)\)-paracompact w.r.t \(\mu\) iff \((X_i, \tau_i, \mu_i)\) is \(m(\tau_i, \mu_i)\)-paracompact w.r.t \(\mu_i\) for every \(i\).

Proof

The "only if" part. Since \(X = \bigcup_{j \in I} X_j\) is \(\tau\)-closed, then the subspace \((X_i, \tau_i, \mu_i)\) is \(m(\tau_i, \mu_i)\)-paracompact w.r.t \(\mu_i\) for every \(i\).

The "if" part. Let \(U = \{U_\lambda : \lambda \in \Delta\}\) be a \(\tau\)-open cover of \(X\) with \(\text{card. } \leq m\). The collection \(\prod = \{U_\lambda \cap X_i : \lambda \in \Delta\}\) is a \(\tau_i\)-open cover of \(X_i\) with \(\text{card. } \leq m\) for every \(i\).

Since \((X_i, \tau_i, \mu_i)\) is \(m(\tau_i, \mu_i)\)-paracompact w.r.t \(\mu_i\), there exist a \(\mu\)-open refinement \(A_i = \{A_{i, \lambda} : \lambda \in \Delta\}\) of \(\prod\) which is locally finite w.r.t \(\mu_i\).

Let \(W = \left\{ \bigcup_{\lambda \in \Delta} A_{i, \lambda} \right\}\).

Then \(W\) is \(\mu\)-open cover of \(X\) refining \(U\), and locally finite w.r.t \(\mu\).

Hence \((X, \tau, \mu)\) is \(m(\tau, \mu)\)-paracompact w.r.t \(\mu\).

2.23 - Corollary

Let \((X, \tau, \mu)\) be a bitopological space and \(\mathcal{X} = \{X_i : X_i \in \tau \cap \mu : i \in I\}\) be a partition of \(X\). The space \((X, \tau, \mu)\) is \((\tau, \mu)\)-paracompact w.r.t \(\mu\) iff the space \((X_i, \tau_i, \mu_i)\) is \((\tau_i, \mu_i)\)-paracompact w.r.t \(\mu_i\) for every \(i\).

2.24 - Theorem

Let \((X, \tau, \mu)\) be a \(m(\tau, \mu)\)-paracompact w.r.t \(\mu\) bitopological space and let \((Y, \tau_Y, \mu_Y)\) be a subspace of \((X, \tau, \mu)\). If \(Y\) is \(F_\alpha\)-set relative to \(\tau\) then \((Y, \tau_Y, \mu_Y)\) is \(m(\tau_Y, \mu_Y)\)-semiparacompact w.r.t \(\mu_Y\).

Proof

Suppose \(Y\) is \(F_\alpha\)-set relative to \(\tau\). Then \(Y = \bigcup Y_n\) where each \(Y_n\) is \(\tau\)-closed. Let \(U = \{U_\lambda : \lambda \in \Delta\}\) be a \(\tau_Y\)-open cover of \(Y\) with \(\text{card. } \leq m\). Since each \(U_\lambda\) is \(\tau_Y\)-open
subset of Y, we have $U_\lambda = V_\lambda \cap Y$, where $V_\lambda$ is $\tau$-open subset of X for each $\lambda \in \Delta$. For each fixed $n$, $E_n = \{V_\lambda : \lambda \in \Delta\} \cup \{X/Y\}$ form a $\tau$-open cover of X with card. $\leq m$. By hypothesis $E_n$ has a $\mu$-open refinement $W = \{W_{\lambda,n} : (\lambda,n) \in \Delta \times IN\}$ which is locally finite. Let $B_n = \{W_{\lambda,n} \cap Y : W_{\lambda,n} \cap Y \neq \emptyset\}$. Let $B = \bigcup B_n$, then B is collection of $\mu_Y$-open sets, covers Y refines U and $\sigma$-locally finite w.r.t. $\mu_Y$. There for $(X, \tau, \mu)$ is $(\tau_Y - \mu_Y)$ semiparacompact w.r.t. $\mu_Y$.

2.25 - Corollary

Let $(X, \tau, \mu)$ be a $(\tau - \mu)$ paracompact w.r.t. $\mu$ biological space and let $(Y, \tau_Y, \mu_Y)$ be a subspace of $(X, \tau, \mu)$. If Y is $F\sigma$-set relative to $\tau$ then $(Y, \tau_Y, \mu_Y)$ is $(\tau_Y - \mu_Y)$ semiparacompact w.r.t. $\mu_Y$.

2.26 - Corollary

Let $(X, \tau, \mu)$ be a $(\tau - \mu)$ paracompact w.r.t. $\mu$ bitopological space and let $(Y, \tau_Y, \mu_Y)$ be a subspace of $(X, \tau, \mu)$. If Y is $F\sigma$-set relative to $\tau$ then $(Y, \tau_Y, \mu_Y)$ is $(\tau_Y - \mu_Y)$-a-paracompact w.r.t. $\mu_Y$.

2.27 - Corollary

Let $(X, \tau, \mu)$ be a $(\tau - \mu)$ paracompact w.r.t. $\mu$ bitopological space and let $(Y, \tau_Y, \mu_Y)$ be a subspace of $(X, \tau, \mu)$. If Y is $F\sigma$-set relative to $\tau$, then $(Y, \tau_Y, \mu_Y)$ is $(\tau_Y - \mu_Y)$ semiparacompact w.r.t. $\mu_Y$.

2.28 - Theorem

Let $(X, \tau, \mu)$ be a bitopological space and let $(Y, \tau_Y, \mu_Y)$ be a $\tau$-closed subspace of $(X, \tau, \mu)$. If $(X, \tau, \mu)$ is $m(\tau-\mu)$-a-paracompact w.r.t. $\mu$, then $(Y, \tau_Y, \mu_Y)$ is $m(\tau_Y, \mu_Y)$-a-paracompact w.r.t. $\mu_Y$.

Proof

Suppose that. $(Y, \tau_Y, \mu_Y)$ be a $\tau$-closed subspace of $(X, \tau, \mu)$. To show that $(Y, \tau_Y, \mu_Y)$ is $m(\tau_Y - \mu_Y)$-a-paracompact w.r.t. $\mu_Y$.

Let $U = \{U_\lambda : \lambda \in \Delta\}$ be a $\tau_Y$-open cover of Y with card. $\leq m$. Since each $U_\lambda$ is a $\tau_Y$-open subset of Y, there is a $\tau$-open subset $V_\lambda$ of X such that each the collection

\[ \Pi = \{V_\lambda : \lambda \in \Lambda\} \cup \{X/Y\} \] form a \( \tau \)-open cover of \( X \) with card. \( \leq m \). By hypothesis \( \Pi \) has refinement \( W = \{W_\gamma : \gamma \in \Gamma\} \) (not necessarily either \( \mu \)-open or \( \mu \)-closed) which is locally finite w.r.t \( \mu \).

Now, let \( A = \{W_\gamma \cap Y, \gamma \in \Gamma\} \) then \( A \) is a collection of subsets of \( Y \) (not necessarily either \( \mu_Y \)-open or \( \mu_Y \)-closed). Then \( A \) is a cover \( Y \) refines \( U \) and is locally finite w.r.t \( \mu_Y \). Therefore \( (X, \tau_Y, \mu_Y) \) is \( m(\tau_Y - \mu_Y) \)-a-paracompact w.r.t \( \mu_Y \).

2.29 - Corollary

Let \( (X, \tau, \mu) \) be a bitopological space and let \( (Y, \tau_Y, \mu_Y) \) be a \( \tau \)-closed subspace of \( (X, \tau, \mu) \). If \( (X, \tau, \mu) \) is \( (\tau - \mu) \)-a-paracompact w.r.t \( \mu \), then \( (Y, \tau_Y, \mu_Y) \) is \( (\tau_Y - \mu_Y) \)-a-paracompact w.r.t \( \mu_Y \).

2.30 - Theorem

Let \( (X, \tau, \mu) \) be a bitopological space and let \( \chi = \{X_i : X_i \in \tau \cap \mu : i \in I\} \) be a partition of \( X \). The space \( (X, \tau, \mu) \) is \( m(\tau - \mu) \)-a-paracompact w.r.t \( \mu \) iff \( (X_i, \tau_i, \mu_i) \) is \( m(\tau_i - \mu_i) \)-a-paracompact w.r.t \( \mu_i \) for every \( i \).

Proof

The "only if" part. Since
\[ X_i = X / \bigcup_{j \neq i} X_j \]
is \( \tau \)-closed, then the subspace \( (X_i, \tau_i, \mu_i) \) is \( m(\tau_i - \mu_i) \)-a-paracompact w.r.t \( \mu_i \) for every \( i \).

The "if" part.

Let \( U = \{U_\lambda : \lambda \in \Delta\} \) be a \( \tau \)-open cover of \( X \) with card. \( \leq m \). The collection \( \Pi = \{U_\lambda \cap X_i : \lambda \in \Delta\} \) is a \( \tau_i \)-open cover of \( X_i \) with card. \( \leq m \) for every \( i \). \( (X_i, \tau_i, \mu_i) \) is \( m(\tau_i - \mu_i) \)-a-paracompact w.r.t \( \mu_i \) \( \forall i \), there exist a refinement \( A_i = \{A_{i,\lambda} : \lambda \in \Delta \} \) of \( \Pi \) (not necessarily either \( \mu_i \)-open or \( \mu_i \)-closed) which is locally finite w.r.t \( \mu_i \).

Let \( W = \{ \bigcup_{i \in I} A_{i,\lambda} : \lambda \in \Delta \} \).
Then \( W \) is a cover of \( X \) (not necessarily either \( \mu \)-open or \( \mu \)-closed), refine \( U \) and is locally finite w.r.t \( \mu \). Hence \( W \) locally finite w.r.t \( \mu \). Hence \( (X, \tau, \mu) \) is a \( m(\tau - \mu) \)-a-paracompact w.r.t \( \mu \).

2.31 - Corollary

Let \((X, \tau, \mu)\) be a bitopological space and let \(\mathcal{X} = \{ X_i : X_i \in \tau \cap \mu \in I \}\) be a partition of \(X\). The space \((X, \tau, \mu)\) is \((\tau, \mu)\)-a-paracompact w.r.t. \(\mu\) iff the space \((X_i, \tau_i, \mu_i)\) is \((\tau_i, \mu_i)\)-a-paracompact w.r.t. \(\mu_i\) for every \(i\).

2.32 - Theorem

If each \(\tau\)-open set in an \(m(\tau, \mu)\) paracompact w.r.t. \(\mu\) bitopological space \((X, \tau, \mu)\) is \(m(\tau, \mu)\) paracompact w.r.t. \(\mu\), then every subspace \((Y, \tau_Y, \mu_Y)\) is \(m(\tau_Y, \mu_Y)\) paracompact w.r.t. \(\mu_Y\).

Proof

Let \(U = \{U_\lambda : \lambda \in \Delta\}\) is a \(\tau_Y\)-open cover of \(Y\) with \text{card.} \leq m\). Since each \(U_\lambda\) is \(\tau_Y\)-open in \(Y\), we have \(U_\lambda = V_\lambda \cap Y\) where \(V_\lambda\) is a \(\tau\)-open subset of \(X\), for every \(\lambda \in \Delta\). Then \(G = \bigcup_{\lambda \in \Delta} V_\lambda\) is a \(\tau_Y\)-open set. Let \(V = \{V_\lambda : \lambda \in \Delta\}\) be a \(\tau_Y\)-open cover of \(G\) with \text{card.} \leq m\). By hypothesis \(G\) is \(m(\tau, \mu)\) paracompact w.r.t. \(\mu\). Thus \(V\) has a \(\mu\)-open refinement \(A = \{A_\gamma, \gamma \in \Gamma\}\) which is locally finite w.r.t. \(\mu\).

Set \(B = \{B_\gamma, \gamma \in \Gamma\}\), where \(B_\gamma = A_\gamma \cap Y\).

Then \(B\) is \(\mu_Y\)-open cover of \(Y\), refine \(U\), and locally finite w.r.t. \(\mu_Y\).

Therefore \((Y, \tau_Y, \mu_Y)\) is \(m(\tau_Y, \mu_Y)\) paracompact w.r.t. \(\mu_Y\).

2.33 - Corollary

If each \(\tau\)-open set in \((\tau, \mu)\) paracompact w.r.t. \(\mu\), the bitopological space is \((\tau, \mu)\) paracompact w.r.t. \(\mu\). Then every subspace \((Y, \tau_Y, \mu_Y)\) is \((\tau_Y, \mu_Y)\) paracompact w.r.t. \(\mu_Y\).

2.34 - Theorem

If \(f\) is \((\mu, \tau)\) closed, \((\mu, \mu^\prime)\) continuous mapping of a bitopological space \((X, \tau, \mu)\) onto \(m(\tau, \mu^\prime)\) paracompact w.r.t. \(\mu^\prime\) bitopological space \((Y, \tau^\prime, \mu^\prime)\) such that \(Z = f^{-1}(y)\) is \(m(\tau, \mu)\) compact, then \((X, \tau, \mu)\) is \(m(\tau, \mu)\) paracompact w.r.t. \(\mu\).
**Proof**

Let \( U = \{ U_\lambda : \lambda \in \Delta \} \) be a \( \tau \)-open cover of \( X \) with \( \text{card.} \leq m \). Then \( U \) is a cover of \( Z \).

Since \( Z \) is \( m(\tau,\mu) \) compact, there exists a finite subset \( \gamma \) of \( \Delta \) such that \( Z \subset \bigcup_{\lambda \in \gamma} U_\lambda \), where \( U_\lambda \) is a \( \mu \)-open set for every \( \lambda \in \gamma \).

Let \( \Gamma \) be the family of all finite sub set \( \gamma \) of \( \Delta \), then \( |\Gamma| \leq m \).

Set
\[
V_\gamma = Y / f \left[ X / \bigcup_{\lambda \in \gamma} U_\lambda \right] .
\]

Since \( \bigcup_{\lambda \in \gamma} U_\lambda \) is a \( \mu \)-open set , the set \( X / \bigcup_{\lambda \in \gamma} U_\lambda \) is \( \mu \)-closed , and since \( f \) is \( (\mu,\tau') \) closed , then \( f \left[ X / \bigcup_{\lambda \in \gamma} U_\lambda \right] \) is \( \tau' \)-closed in \((Y,\tau',\mu')\) , hence \( V_\gamma \) is a \( \tau' \)-open .

Moreover \( y \in V_\gamma \) and \( f^{-1} \left[ V_\gamma \right] \subset \bigcup_{\lambda \in \gamma} U_\lambda \). Therefore \( V = \{ V_\gamma : \gamma \in \Gamma \} \) is a \( \tau' \)-open cover of \( Y \) with \( \text{card.} \leq m \). Since \((Y,\tau',\mu')\) is \( m(\tau',\mu') \) paracompact w.r.t. \( \mu' \), then \( V \) has a \( \mu' \)-open refinement \( W = \{ W_\delta : \delta \in \Omega \} \) which is locally finite w.r.t. \( \mu' \). Set \( \Pi = \{ f^{-1} \left[ W_\delta \right] \cap U_\lambda : (\delta,\lambda) \in \Omega \times \gamma \} \); then \( \Pi \) is a \( \mu \)-open cover of \( X \), refines \( U \), and locally finite w.r.t. \( \mu \). Therefore \((X,\tau,\mu)\) is \( m(\tau,\mu) \) paracompact w.r.t. \( \mu \).

**2.35 - Corollary**

If \( f \) is \( (\mu,\tau') \) closed , \( (\mu,\mu') \) continuous mapping of a bitopological space \((X,\tau,\mu)\) onto \((\tau',\mu') \) paracompact w.r.t. \( \mu' \) bitopological space \((Y,\tau',\mu')\) such that \( Z = f^{-1}(y) : y \in Y \) is \( (\tau,\mu) \) compact , then \((X,\tau,\mu)\) is \((\tau,\mu) \) paracompact w.r.t. \( \mu \).

**2.36 - Theorem**

If \( f \) is \( (\mu,\tau') \) closed , \( (\mu,\mu') \) continuous mapping of a bitopological space \((X,\tau,\mu)\) onto \( m(\tau',\mu') \) semiparacompact w.r.t. \( \mu' \) bitopological space \((Y,\tau',\mu')\) such that \( Z = f^{-1}(y) : y \in Y \) is \( m(\tau,\mu) \) compact , then \((X,\tau,\mu)\) is \( m(\tau,\mu) \) semiparacompact w.r.t. \( \mu \).

**Proof**

Let \( U = \{ U_\lambda : \lambda \in \Delta \} \) be a \( \tau \)-open cover of \( X \) with \( \text{card.} \leq m \). Then \( U \) is a cover of \( Z \). Since \( Z \) is \( m(\tau,\mu) \) compact, there exists a finite subset \( \gamma \) of \( \Delta \) such that

\[
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\]
That \( Z \subseteq \bigcup_{\lambda \in \gamma} U_{\lambda} \), where \( U_{\lambda} \) is a \( \mu \)-open set for every \( \lambda \in \gamma \). Let \( \Gamma \) be the family of all finite subset \( \gamma \) of \( \Delta \), then \( |\Gamma| \leq m \).

Set
\[
V_\gamma = Y / f \left( X / \bigcup_{\lambda \in \gamma} U_{\lambda} \right).
\]
Since \( \bigcup_{\lambda \in \gamma} U_{\lambda} \) is \( \mu \)-open set, the set \( X / \bigcup_{\lambda \in \gamma} U_{\lambda} \) is \( \mu \)-closed, and since \( f \) is \( (\mu, \tau) \)-closed, then \( f \left[ X / \bigcup_{\lambda \in \gamma} U_{\lambda} \right] \) is \( \tau \)-closed in \((Y, \tau, \mu')\) hence \( V_\gamma \) is \( \tau \)-open and \( \gamma \in V_\gamma \) and \( f^{-1}[V_\gamma] \subseteq \bigcup_{\lambda \in \gamma} U_{\lambda} \). Therefore \( V = \{V_\gamma : \gamma \in \Gamma\} \) is a \( \tau \)-open cover of \( Y \) with card. \( \leq m \). Since \((Y, \tau, \mu')\) is \( m \) \((\tau, \mu')\) semiparacompact w. r. t \( \mu' \), then \( V \) has a \( \mu' \)-open refinement \( W = U_n W_n \) where every \( W_n \) is locally finite w. r. t \( \mu' \).

Set
\[
W_n = \{W_{n, \delta} : \delta \in \Omega\}. \quad \text{Thus} \quad W = U_n \{W_{n, \delta} : \delta \in \Omega\}.
\]
Set \( C_n = U_n C_n \), where \( C_n = \left\{ f^{-1}[W_{n, \delta}] \cap U_{\lambda} : (\delta, \lambda) \in \Omega \times \gamma_\delta \right\} \). We claim that \( C_n \) is
(i) collection of \( \mu \)-open sets;
(ii) locally finite w. r. t. \( \mu \);

Proof of (i)

Since \( W_{n, \delta} \) is a \( \mu' \)-open \( \forall \delta \in \Delta \) and \( f \) is \( (\mu, \mu') \) continuous, the set \( f^{-1}[W_{n, \delta}] \) is a \( \mu \)-open \( \forall \delta \in \Delta \), and since \( U_{\lambda} \) is a \( \mu \)-open \( \forall \lambda \in \gamma_\delta \), then \( f^{-1}[W_{\delta}] \cap U_{\lambda} \) is a \( \mu \)-open \( \forall (\delta, \lambda) \in \Delta \times \gamma_\delta \).

Proof of (ii)

Let \( x \in X \Rightarrow \exists y \in Y \ni y = f(x) \). Since \( W_n \) is locally finite w. r. t \( \mu \Rightarrow \exists \mu_y \neg \text{nhd of } x \) such that \( N \cap W_n = \emptyset \) for all but finite number of \( \delta \Rightarrow f^{-1}[\emptyset] \cap \left( f^{-1}[W_{n, \delta}] \cap U_{\lambda} \right) = \emptyset \) for all but finite number of \( (\delta, \lambda) \) since \( f \) is \( (\mu, \mu') \) continuous, then \( f^{-1}[\emptyset] \) is a \( \mu \)-nhd of \( x \). Hence \( C_n \) is locally finite w. r. t \( \mu \). Its remains to show that \( C \) is:

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(i*) cover X , and 
(ii*) refine U
proof of (i*)

Let $x \in X \Rightarrow \exists U_{\lambda} \ni x \in U_{\lambda}$ and $\exists y \in Y \exists y = f(x) \Rightarrow \exists W_{n,\delta} \ni y \in W_{n,\delta}$ for some $n, \delta \Rightarrow x \in f^{-1}[W_{n,\delta}]$ for some $n, \delta \Rightarrow x \in f^{-1}[W_{n,\delta}] \cap U_{\lambda}$ for some $(\delta, \lambda)$.

Proof of (ii*)

Since $f^{-1}[W_{n,\delta}] \cap U_{\lambda} \subset U_{\lambda}$, $\forall n, \delta \Rightarrow \bigcup_{n=1}^{\infty} f^{-1}[W_{n,\delta}] \cap U_{\lambda} \subset U_{\lambda}$

i.e $\Pi$ refine $U_{\lambda}$. Therefore $(X, \tau, \mu)$ is $m(\tau-\mu)$ semiparacompact w. r. t $\mu$.

2.37 - Corollary

If $f$ is ($\mu$- $\tau$) closed , ($\mu$-$\mu$') continuous mapping of a bitopological space $(X, \tau, \mu)$ onto $(\tau$-$\mu$')semiparacompact w.r.t.$\mu$' bitopological space $(Y, \tau, \mu')$ such that $Z= f^{-1}(y) ; y \in Y$ is ($\tau$-$\mu$) compact , then $(X, \tau, \mu)$ is ($\tau$-$\mu$)semiparacompact w. r. t. $\mu$.

References
