Some Geometric Properties of Julia Sets of Maps Of The Form $(\lambda z - \lambda z^2)$

Naoum-Adil,Department of Mathematics, College of Science, Baghdad University.
Talb-Iftichar Department of Mathematics, College of Education, Babylon University.
M. And Al.salami-H.Q,Department of Mathematics, College of Education, Babylon University

Abstract

In this work, we will study the geometric properties of Julia sets of the quadratic polynomial maps of the form $(\lambda z - \lambda z^2)$ *where* λ *is a non-zero complex .We show that Julia set is the unit circle if* λ *=2 and Julia set is the line segment if* λ *=4 . If* $1<|\lambda|<1+\sqrt{2}$ *. Then the Julia set is a simple closed curve ,also if* $1<|\lambda|<1+\sqrt{2}$ *then the Julia set is a simple closed curve such that Julia set which contains no smooth arcs, and if* $\lambda = 1 \mp \sqrt{5}$ *then the Julia set is infinitely many different simple closed curves.*

Introduction

In complex dynamics , the iteration theory originated in 1910 [7] . Among the most important concepts in complex dynamics are Julia sets .They were studied by the French mathematician Gaston Julia (1893 – 1978) , who developed much of theory when he was recovering from his wounds in an army hospital during world war I . He published a long paper in French language in [4],Julia and Fatou looked at the iteration of the simplest quadratic map of the form $(z^2 + c)$. In general , distinct maps have distinct Julia sets , however , there exist distinct polynomial maps , rational maps and entire maps that have the same Julia sets [5], [6].The Julia set of a polynomial typically has a complicated , self – similar structure. Therefore the Julia sets are fractals [2] ,[7] .However , there exist rational maps whose Julia sets fail to be quasi-self-similar [3] .

1 - Preliminary Definitions

Let C be the complex set or complex plane. The complex plane together with the point at infinity, denoted by ∞ , is called the extended complex plane, it is topologically equivalent to the Riemann sphere. We put $C_\infty = C \cup \{\infty\}$ .The metric space of the complex plane is the usual metric , while the metric space of the Riemann sphere is the chordal metric .we use the symbol f^n to denote n-th iteration for $n \in N$, f : C $\to$ C is smooth , if f is a C^r - diffeomorphism if f is a C^r-homeomorphism such that f^{-1} is also C^r . A point $x \in X$ is called a fixed point if $f(x) = x$. It is a periodic with period n if $f^n(x) = x$, but $f^m(x) \neq x$ for $m < n$ .. Let x be a periodic point of period n for f . The point x is hyperbolic if $\left|(f^n)'(x)\right| \neq 1$, x is attracting periodic point if $\left|(f^n)'(x)\right| < 1$ and x is repelling periodic point if $\left|(f^n)'(x)\right| > 1$.

Remark (1-1)

The fixed points of $Q_\lambda(z) = \lambda z - \lambda z^2$ are $z = 0$ or $z = \frac{\lambda - 1}{\lambda}$. If if $z=0$ then $\left|Q'_{\lambda}(0)\right|=|\lambda|$. If $|\lambda|$ < 1 , then $z=0$ is attracting fixed point . If $|\lambda|$ > 1 , then $z=0$ is repelling fixed point .If $z=\frac{\lambda-1}{\lambda}$ then $\left|Q'_{\lambda}\left(\frac{\lambda-1}{\lambda}\right)\right|=|2-\lambda|$. If 3 < $|\lambda|$ or $|\lambda|$ < 1, then $z=\frac{|\lambda|-1}{|\lambda|}$ is repelling fixed point . If 1 < $|\lambda|$ < 3 , then $z=\frac{|\lambda|-1}{|\lambda|}$ is attracting fixed point . The critical point for Q_{λ} is 0.5 .

Definition (1-2) [1]

The family $\{f_n\}$ is said to be normal on U if every sequence of the f_n `s has a subsequence which either

1.converges uniformly on compact subsets of U ,or

2. converges uniformly to ∞ on U

Now, we will give the definition of the Fatou set and Julia set :

Definition (1-3) [8]

Let $f : C \to C$ be a map . The Fatou set (stable set), F(f) is the set of points $z\in C$ such that the family of iterates $\{f^n\}$ is normal family in some neighborhood of z.The Julia set $J(f)$ is the complement of the Fatou set , that is $J(f)=\{$ $z\in C$: the family $\{f^n\}_{n\geq 0}$ is not normal at z $\}$ That is $J(f)\equiv C \setminus F(f)$.

Also the previous definition can satisfy on the space C_∞ .

Definition (1-5)[2]

Let $f : C_\infty \to C_\infty$ be a polynomial of degree $n\geq 2$. Let $K(f)$ denote the set of points in C whose orbits do not converge to the point at infinity .

That is $K(f)=\{$ $z\in C:\left\{\left|f^n(z)\right|\right\}_{n=0}^{\infty}$ is bounded $\}$. This set is called filled Julia set

Definition (1-6) [2]

Let $f : C_\infty \to C_\infty$ be a map . The escape set $A(\infty)$ of f is all those points that escape to infinity , that is $A(\infty) = \{ z : f^n(z) \to \infty$ as $n \to \infty \}$.

We can say that $A(\infty)$ is the basin of attraction of ∞ . Now we can state another definition for Julia set .

Definition (1-7) [2]

The Julia set is the boundary of the filled Julia set , that is $J(f) = \partial K(f)$. The complement of the basin of attraction of ∞ is the filled Julia set of f . That is $C_\infty \setminus A(\infty) = K(f)$.

3- Some Examples Of Julia Sets

We will put in this section two examples to find the Julia sets :

Example (3-1)

$J(Q_2)$ is the unit circle of $Q_2(z) = 2z - 2z^2$. The discussion of this example splits into three claims .

Let $D(a,b) = \{ z \in C : |z - a| < b \}$, where $a \in C$ and $0 < b \in R$.

Claim 1 : Let $z_0 \in D(0,1)$, then $z_0 \in F(Q_2)$. Let $z_0 \in D(0,1)$, that is $|z_0| < 1$. Suppose that $U = D\left(z_0, \frac{1 - |z_0|}{2} \right)$. One can see that $U \subseteq D(0,1)$ for all $z \in \overline{U}$ and by using $|z - z_0| \geq |z| - |z_0|$, thus $|z - z_0| < \frac{1 - |z_0|}{2}$, hence $|z| - |z_0| \leq |z - z_0| < \frac{1 - |z_0|}{2}$, therefore $|z| - |z_0| < \frac{1 - |z_0|}{2}$, thus $|z| < \frac{1}{2} - \frac{|z_0|}{2} + |z_0|$,

hence $|z| < \frac{1}{2} + \frac{|z_0|}{2}$, that is for all $z \in \overline{U}$, $|z| < \frac{1 + |z_0|}{2} < 1$. Hence $\overline{U} \subset D(0,1)$.

For

all $z \in \overline{U}$, $Q_2(z) = 2z - 2z^2$,if $|Q_2(z)| = |2z - 2z^2| \leq |2z| + |2z^2| < 2|z^2| + 2|z^2|$

$=4|z^2|$, thus $\left|Q_2^2(z)\right| = |4z-12z^2+16z^3-8z^4| \le |4z|+|12z^2|+|16z^3|+|8z^4|$ $< 16|z^4|+16|z^4|+16|z^4|+16|z^4| = 4^3|z^4|$,

hence for n -th iterate $\left|Q_2^n(z)\right| \to 0$ as $n\to\infty$ .Therefore $\{Q_2^n\}$ is normal in $D(0,1)$, hence $D(0,1)\subseteq F(Q_2)$.

Claim2 :If $|z_0|>1$, then $z_0\in A_2(\infty)$ .Let $|z_0|>1$. $\left|Q_2(z_0)\right| = |2z_0-2z_0^2| \le 2|z_0|+2|z_0|^2 < 2|z_0^2|+2|z_0^2| = 4|z_0^2|$,Then $\left|Q_2^2(z_{0z})\right| = |4z_0-12z_0^2+16z_0^3-8z_0^4| \le |4z_0|+|12z_0^2|+|16z_0^3|+|8z_0^4|$ $<$ $16|z_0^4|+16|z_0^4|+16|z_0^4|+16|z_0^4| = 4^3|z_0^4|$ .Hence , for n -th as $n\to\infty$. Therefore $z_0\in A_2(\infty)$.

Claim 3: If $|z_0|=1$, then $z_0\notin F(Q_2)$ and $z_0\notin A_2(\infty)$. Let $|z_0|=1$. Assume $z_0\in F(Q_2)$ so there exists neighborhood U_{z_0} , which has a subsequence of $\{Q_2^n\}$, and a map f with $Q_2^{n_k}\to f$ uniformly on U_{z_0} . Now for all $\varepsilon > 0$ there is $D(z_0,\varepsilon)\subset U_{z_0}$, by claim 1 , there is $z_1\in D(z_0,\varepsilon)$ with $|z_1| < 1$. It follows that $Q_2^{n_k}\to 0$ as $n\to\infty$, that is $f(z_1)=0$ .Since $|z_0|=1$, $\left|f(z_1)\right|=1$, which is contradicts that f is analytic map (and therefore is continuous). Therefore $z_0\notin F(Q_2)$. Similarly , we can proof that $z_0\notin A_2(\infty)$. Therefore $z_0\in J(Q_2)$ for $|z_0|=1$. Hence $J(Q_2)$ is unit circle . ■

Example (3-2)

$J(Q_4)$ is the line segment [0,1] for $Q_4(z)=4z-4z^2$,the discussion of this example splits into three claims .

Claim 1: The set [0,1] is completely invariant . Consider $Q_4(x)=4x-4x^2$, thus $Q'_4(x)=4-8x$, hence $Q''_4(x)=-8$, therefore $Q_4(x)$ has maximum value 1 at x =0.5 since $Q_4(0.5)=1$. $Q_4(x)$ is increasing on the interval [0.5,1] . $Q_4(x)$ has minimum value of 0 at x = 0 or 1 , since $Q_4(0)=0$ and $Q_4(1)=0$. Thus $Q_4([0,1])\subset[0,1]$ .Therefore $[0,1]$ is not a subset of $A_4(\infty)$ Claim 2: $W=C_\infty\setminus[0,1]$ is $A_4(\infty)$ .Let $z_0\in W$ with $|z_0|>1$ .If

$\left|Q_4(z_0)\right| = \left|4z_0 - 4z_0^2\right| \le 4|z_0| + 4|z_0^2| < 4|z_0^2| + 4|z_0^2| = 8|z_0^2|$,thus

$\left|Q_4^2(z_0)\right| = \left|16z_0 - 80z_0^2 + 128z_0^3 - 64z_0^4\right|$

$\le |16z_0| + |80z_0^2| + |128z_0^3| + |64z_0^4|$

$< 128|z_0^4| + 128|z_0^4| + 128|z_0^4| + 128|z_0^4| = 8^3|z_0^4|$ .Hence , for n-th as $n \to \infty$. Therefore $z_0 \in A_4(\infty)$.

Claim 3 : [0,1] is the Julia set for $Q_4(z) = 4z - 4z^2$,since $[0,1] = \partial A_4(\infty)$. Hence [0,1] is the Julia set for Q_4 . ■

4- Properties of Julia Sets

In this section , we introduce some geometric properties :

Proposition (4-1)

Suppose that $1<|\lambda|<1+\sqrt{2}$. Then $J(Q_\lambda)$ is a simple closed curve .

Proof :

$Q_\lambda(z) = |\lambda|z - |\lambda|z^2$, then $\left|Q'_\lambda(z)\right| = |\lambda - 2\lambda z| < 1$, thus $|\lambda||1-2z| < 1$, that is $|1-2z| < \dfrac{1}{|\lambda|}$, since $|a-b| > |a| - |b|$ thus $1 - 2|z| < \dfrac{1}{|\lambda|}$, hence $|z| > \dfrac{1}{2} - \dfrac{1}{2|\lambda|}$, or $|z| < \dfrac{1}{2} + \dfrac{1}{2|\lambda|}$, thus $|z - 0.5| < \dfrac{1}{2|\lambda|}$, where $\dfrac{1}{2|\lambda|}$ is the radius and 0.5 is the center .We note if $1<|\lambda|<1+\sqrt{2}$ =2.4142135 , then $\dfrac{1}{2|\lambda|}$ < 0.2071067 , $0.5 + \dfrac{1}{2|\lambda|}$ < 0.7071067 and $0.5 - \dfrac{1}{2|\lambda|}$ < 0.2928933 .

The attractor point is $\dfrac{|\lambda| - 1}{|\lambda|}$ < 0.5857864 , while the critical point of Q_λ and the centre of circle is 0.5 . $\left|Q_\lambda(0.5)\right| < 0.6035533$

$\left|Q_\lambda(0.5857864)\right| < 0.5857863$ and $\left|Q_\lambda(0.7071067)\right| < 0.5$ but

$\left|Q_\lambda(0.2928933)\right| < 0.5$, $\left|Q_\lambda(0.5 + 0.2071067i)\right| < 0.7071064$

$\left|Q_\lambda(0.5-0.2071067i)\right| < 0.7071064$ and $\left|Q_\lambda(0)\right| = 0$

$\left|Q_\lambda(2)\right| < 4.828427$,also $\left|Q_\lambda(0.1)\right| < 0.2172792$, $\left|Q_\lambda(0.8)\right| < 0.3862741$.

Let Γ_0 be the circle of radius 0.2071067 about 0.5 . Γ_0 contains both the attracting fixed point (0.5857864) and the critical point 0.5 of Q_λ in its interior . Moreover , $\left|Q'_\lambda(z)\right| > 1$ for z in the exterior of Γ_0, where 0 is repelling fixed point of Q_λ. For each $\theta \in S^1$, we will define a continuous curve $\gamma_\theta : [1,\infty) \to C$ having the property that $z(\theta) = \lim_{t\to\infty} \gamma_\theta(t)$ is a continuous parameterization of $J(Q_\lambda)$ .To define $z(\theta)$, we first note that the preimage Γ_1 of Γ_0 under Q_λ is $Q_\lambda(z) = \lambda z - \lambda z^2 = w$, thus $\lambda^2 - \lambda z + w = 0$, hence $z = \frac{1}{2} \mp \sqrt{\frac{1}{4} - \frac{w}{\lambda}}$ .The preimage with respect to 0.7071067 and 0.2928933 are $z = 0.5 \mp 0.2071069$, that is with respect to 0.7071067 is $z = 0.7071069$ and $z = 0.2928931$, also with respect to 0.2928933 is $z = 0.7071069$ and $z = 0.2928931$, while the preimage with respect to the attracting fixed point (0.5857864) is $z = 0.5 \mp 0.0857869$, that is $z = 0.5857869$ and $z = 0.4142131$, while the preimage with respect to the critical point (0.5) is $z = 0.5003162$ and $z = 0.4996838$, while the preimage for the points with respect to (0.5+0.2071067i) and (0.5-0.2071067i) are $z = 0.5 \mp 0.2071062i$, that is $z = 0.5+0.2071062i$ and $z = 0.5-0.2071062i$ and $z = 0.5+0.2071062i$ and $z = 0.5-0.2071062i$,each value of the preimages under Q_λ have two values, as follows

$\left|Q_\lambda(0.7071069)\right| <$ 0.4999997 ,also $\left|Q_\lambda(0.2928931)\right| <$ 0.4999997, $\left|Q_\lambda(0.5-0.2071062i)\right| <$ 0.7071059 and $\left|Q_\lambda(0.5+0.2071062i)\right| <$ 0.7071059 ,

and the value of the preimages under Q_λ for the critical point , as follows

$\left|Q_\lambda(0.5003162)\right| < 0.603553$, $\left|Q_\lambda(0.4996838)\right| < 0.6035531$.

While the value of the preimages under Q_λ for the attracting fixed point , as follows $|Q_\lambda(0.5857869)| < 0.5857862$ and $|Q_\lambda(0.4142131)| < 0.5857862$.

Then preimage Γ_1 of Γ_0 under Q_λ is a simple closed curve which contains Γ_0 in its interior and which is mapped in a two – to – one formula onto Γ_0 .

The fact that Γ_1 is a simple closed curve follows from the fact that both the critical point (0.5) and its image lie inside Γ_0 . Hence the curves Γ_0 and Γ_1 bound an annular region A_1 (A_1 may be regarded as a fundamental domain for the attracting fixed point for Q_λ) Let W be the standard annulus defined by $W = \{ \; r e^{i\theta} : 1 \le r \le 2, \theta$ arbitrary $\}$ .Choose diffeomorphism $\varphi : W \to A_1$ which maps the inner and outer boundaries of W to the corresponding boundaries of A_1 . See figure (1) . This allows us to define the initial segment of $\gamma_\theta : [1,2] \to C$ by $\gamma_\theta(r) = \varphi(r e^{i\theta})$. That is , γ_θ is the image of a ray in W under φ .

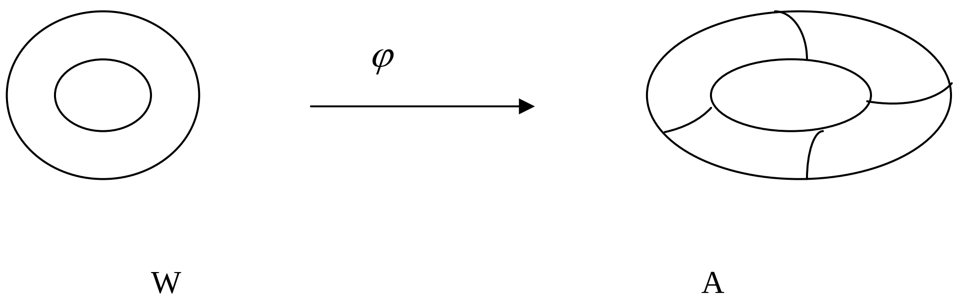

Fig.1

For $r \ge 2$, may extend γ_θ as follows , since preimage Γ_1 of Γ_0 under Q_λ and the critical point in interior Γ_0 , thus Q_λ has no critical points in the exterior of Γ_1 . The preimages Γ_2 of Γ_1 under Q_λ are

$z = 0.5 \mp 0.2071072$, that is $z = 0.7071072$ and $z = 0.2928928$ with respect to 0.7071069 , also have the same preimages with respect to 0.2928931, while the preimages of critical points (0.5003162) and (0.4996838) are $z = 0.5 \mp 0.0004472$

that is $z = 0.5004472$ and $z = 0.4995528$ with respect to 0.5003162 and also for 0.4996838, while the preimages for the attracting fixed points are $z = 0.5 \mp 0.0857875$ that is $z = 0.5857875$ and $z = 0.4142125$ for (0.5857869) and also for (0.4142131) , while the preimages of points (0.5+0.2071062i) and (0.5 -0.2071062i) are $z = 0.5 \mp 0.2071057\, i$ that is $z = 0.5 + 0.2071057\, i$ and $z = 0.5 - 0.2071057\, i$ for (0.5+0.2071062i) and also for (0.5-0.2071062i),each value of the preimages under Q_λ have four values, as follows $\left|Q_\lambda(0.7071072)\right| < 0.4999995$ and $\left|Q_\lambda(0.2928928)\right| < 0.4999995$

$\left|Q_\lambda(0.5 - 0.2071057i)\right| < 0.7071055$,and $\left|Q_\lambda(0.5 + 0.2071057i)\right| < 0.7071055$,

the value of the preimages under Q_λ for the critical point , as follows

$\left|Q_\lambda(0.5004472)\right| < 0.6035528$ and $\left|Q_\lambda(0.4995528)\right| < 0.6035528$.

While the value of the preimages under Q_λ for the attracting fixed point , as follows

$\left|Q_\lambda(0.5857875)\right| < 0.5857859$ and $\left|Q_\lambda(0.4142125)\right| < 0.5857859$.

Hence there is a simple closed curve Γ_2 which is mapped in a two – to – one formula onto Γ_1.Moreover , Q_λ maps the annular region A_2 between Γ_1 and Γ_2 onto A_1 , again in a two –to-one formula . Thus , the preimage of any γ_θ in A_1 is a pair of non – intersection curves in A_2, thus every point $z \in A_2$, imply $f(z) \in A_1$. There is a unique such curve which meets the inner boundary Γ_1 . Hence , for each θ , there is a unique curve in A_2 which contains the point $\gamma_\theta(2)$, that is $\gamma_\theta(1)$ is boundary of Γ_0 and $\gamma_\theta(2)$ is boundary of Γ_1 and $\gamma_\theta(3)$ is boundary of Γ_2 . We may thus sew together these two curves in the obvious way at this point , producing a single curve defined on the interval [1,3] .Continuing in this formula , we may extend each γ_θ over the entire interval $[0,\infty)$. Now recall that $\left|Q'_\lambda(z)\right| > k > 1$ for positive integer k provided z lies in the exterior of Γ_1 . Hence the length of each extension of γ_θ decreases geometrically . It follows that $\gamma_\theta(t)$ converges uniformly in θ and that $\lim_{t\to\infty} \gamma_\theta(t) = z(\theta)$, since $\lim_{t\to\infty} \gamma_\theta(t)$ is continuous , thus $z(\theta)$ is continuous and is a unique point in C for each θ .We claim that $z(\theta)$ parameterizes a simple closed curve

in C . To show that the image curve is simple , we must prove that if $z(\theta_1) = z(\theta_2)$, then $z(\theta) = z(\theta_1)$ for all θ with $\theta_1 \le \theta \le \theta_2$, see fig. (3). $z(\theta)$ is a point by substituting $\theta = \theta_1$. However , if this was not the case , the portions of the curves Γ_1 , $\gamma_{\theta_1}(t)$ and $\gamma_{\theta_2}(t)$ would bound a simply connected region containing each $z(\theta)$ in its interior . This implies that there is a neighborhood of $z(\theta)$ whose images under Q_λ^n remains bounded , thus $z(\theta)$ is attracting but not repelling .
Hence $z(\theta) \notin J(Q_\lambda)$ .But this is impossible . Therefore $J(Q_\lambda)$ is simple closed curve . ■

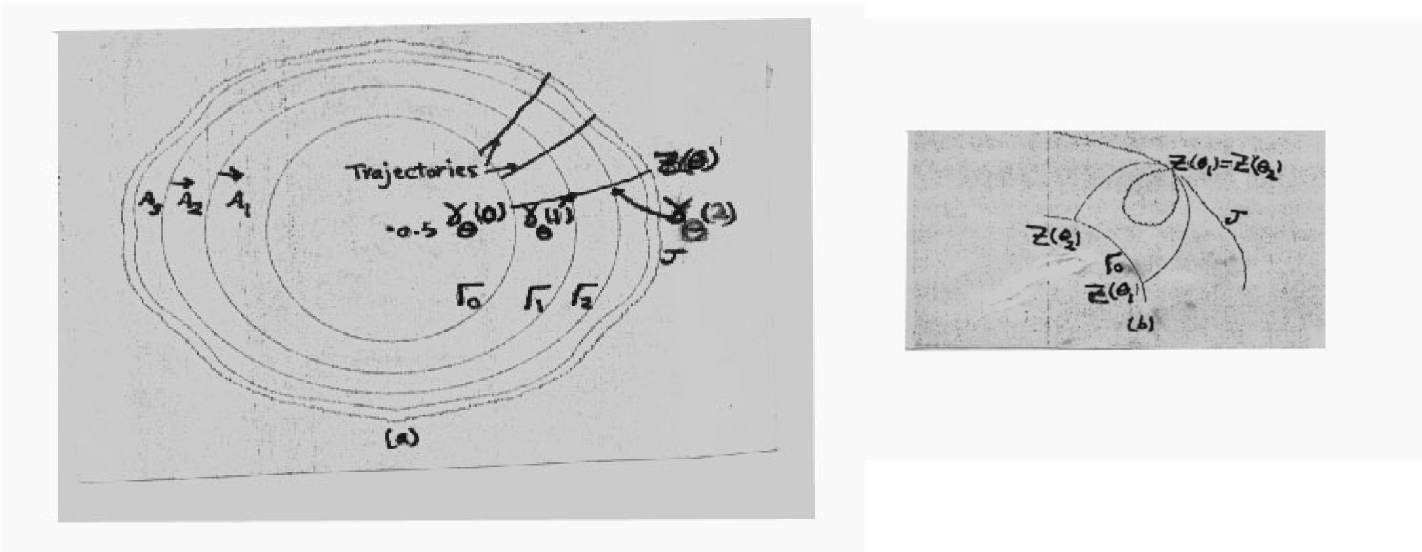

Fig.2 (a) & (b) the proof of the proposition (for $1<|\lambda| < 1+\sqrt{2}$)

Proposition (4-2)

Suppose λ is a complex number and $1<|\lambda|<1+\sqrt{2}$ .Then $J(Q_\lambda)$ is a simple closed curve such that Julia set which contains no smooth arcs .

Proof :

Suppose that λ is complex , that is $\lambda = \lambda_1 + \lambda_2 i$ and satisfies $1<|\lambda|<1+\sqrt{2}$ .If Q_λ has repelling fixed point at $z_0 = 0$. Then $|Q'_\lambda(0)| = |\lambda| - 2|\lambda|(0) = |\lambda|$, if $\lambda_1 \ne 0$ then λ is not pure imaginary , by properties of complex analysis , thus z_0 does not lie in a smooth arc in $z(\theta)$. For if this were the case , then the image of $z(\theta)$ would also be a smooth arc in $J(Q_\lambda)$ passing through z_0 . Since $Q'_\lambda(z_0)$ is complex , the tangents to these two curves $z(\theta_1)$ and $z(\theta_2)$ would not be parallel .Therefore $z(\theta)$ would not be simple at z_0, that is $z(\theta_1) \neq z(\theta_2)$. The preimage of z_0 are dense in $J(Q_\lambda)$ .It follows that $J(Q_\lambda)$ contains no smooth arcs . ■

Example (4-3)

$J(Q_\lambda)$ is infinitely many different simple closed curves for $\lambda = 1 \mp \sqrt{5}$.

First , let $\lambda = 1+\sqrt{5}$. We now turn to the case of an attracting periodic rather than fixed point .$Q_\lambda^2(z) = z$, thus $Q_\lambda^2(z) - z = 0$, hence $\lambda^2 z^2 - z(\lambda^2+\lambda) + (\lambda+1) = 0$, therefore $z = \frac{\lambda+1}{2\lambda} \mp \frac{1}{2\lambda}\sqrt{\lambda^2 - 2\lambda - 3}$, thus $z = 0.5$ and $z = 0.809017$, which $Q_\lambda(0.5) = 0.809017$ and $Q_\lambda(0.809017) = 0.5$. Also $Q'_\lambda(z) = \lambda - 2\lambda z$, thus $|Q'_\lambda(0.5)| = 0 < 1$ is an attracting fixed point .Therefore 0.5 and 0.809017 lie on an attracting periodic of period 2 .The dynamics of Q_λ on the real line relatively straight forward , there are two repelling fixed points at 0 and 0.6909829 , since Q_λ as two repelling fixed point $z = 0$ or $z = \frac{\lambda-1}{\lambda} = 0.6909829$, that is $|Q'_\lambda(0)| > 1$ and $|Q'_\lambda(0.699829)| > 1$. The fixed point at 0.6909829 is the dividing point between the basin of attraction of 0.5 and 0.809017 . By proposition (4-1) , one may show that there are two simple closed curves γ_0 and γ_1 in $J(Q_\lambda)$ which surround 0.5 and 0.809017 respectively .The curves γ_0 and γ_1 meet at fixed point 0.6909829 . There is much more $J(Q_\lambda)$ however . The basin of attraction of 0.5 is not completely invariant because one preimage of the interior of γ_0 is γ_1 but there is another surrounding the other preimage of 0.5 is 0.190983 , since $Q_\lambda(z) = 0.5$, thus $3.2360679z^2 - 3.2360679z + 0.5 = 0$, hence $z = 0.809017$ and $z = 0.190983$. Therefore $Q_\lambda(0.190983) = 0.5$. Hence there is a third simple closed curve in $J(Q_\lambda)$ surrounding 0.190983 as well . Now both

0.190983 and 0.809017 must have a pair of distinct preimages , each is surrounded by a simple closed curve in $J(Q_\lambda)$. Continuing in this formula , we get that the Julia set of Q_λ must contain infinitely many different simple closed curves .In the same way if $\lambda = 1-\sqrt{5}$ then $z = 0.4999998$ and $z = -0.309017$, $Q_\lambda(0.4999998) = -0.309017$ and $Q_\lambda(-.0309017) = 0.4999998$, also $|Q'_\lambda(0.4999998)| < 1$, thus -0.309017 and 0.4999998 lie on an attracting periodic of period 2 , also has two repelling at $z = 0$ and $z = 1.809017$. Hence 0 is the dividing point between the basin of attraction of -0.309017 and 0.4999998 . There are two simple closed curves Γ_0 and Γ_1 in $J(Q_\lambda)$ which surrounds 0.4999998 and -0.309017 respectively .So that if $-1.2360679z^2 + 1.2360679z + 0.999998 = 0$, then $z = -0.309017$ and $z = 1.3091017$, also $Q_\lambda(1.309017) = 0.4999998$. Hence there is third simple a closed curve in $J(Q_\lambda)$ surrounding 1.309017 . See fig. (3) .

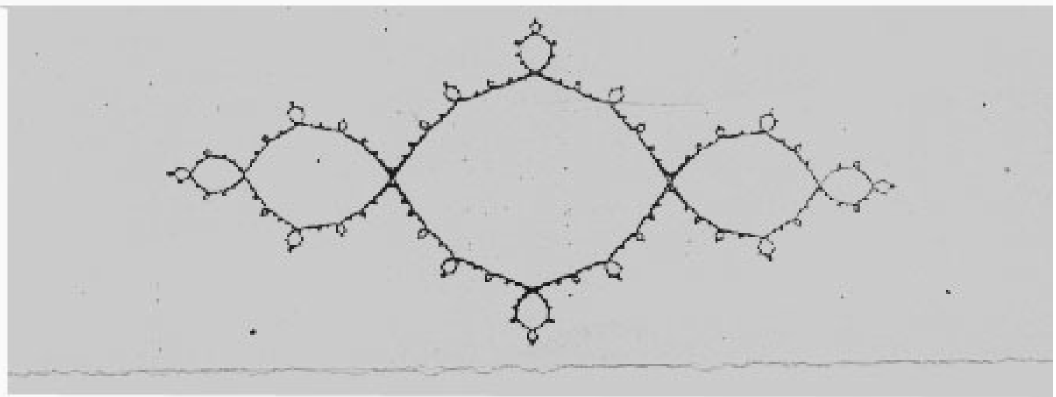

Fig.3 Julia set for $\lambda = 1 \mp \sqrt{5}$.

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