Inverse and direct theorems for monotone approximation

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Abstract

We prove that if \( f \) is increasing function on \([-1,1]\) then for each \( n=1,2,... \), there is an increasing algebraic polynomial \( p_n \) of degree \( 8n \) such that \( \|f - p_n\|_p \leq c(p)\omega_{2}\left( f, \frac{1}{n} \right)_p \),

Where \( \omega_{2} \) is the second order Ditizian - Totik modulus of smoothness. Also a converse theorem for this direct theorem were obtained. These results complement the classical pointwise estimates of the same type for unconstrained polynomial approximation.

1. Introduction and Main Results

Several results show that in some sense monotone approximation by algebraic polynomials performs as well as unconstrained approximation. For example Lorentz and Zeller, [7] have shown that for each increasing function \( f \) in \( C(I) \) ( the
space of all continuous functions on $I = [-1,1]$ there is an increasing polynomial $p_n$ of degree $n$ that satisfies

$$\|f - p_n\| \leq c \omega \left( f, \frac{1}{n} \right), \quad n = 1, 2, \ldots$$

(1.1)

where $\omega$ is the modulus of continuity of $f$.

A general result for (2.1.1) for any $k = 0, 1$, there are increasing $p_n$ that satisfies

$$\|f - p_n\| \leq c n^{-k} \omega \left( f^{(k)}, \frac{1}{n} \right), \quad n = 1, 2, \ldots$$

(1.2)

desirable result of Lorentz [6], where as the general case was proved by DeVore [3], the cases $k = 0, 1$ are much easier to prove than the general cases. Since they can be proved using linear method, in contrast, the proof in [3] uses rather involved non-linear techniques. It is well known that for unconstrained approximation much improvement can be made in estimates of the form (1.1) where $x$ is near the end points of $I$.

In this thesis, we are interested in pointwise estimates for monotone approximation, the only result of this type that we know of is by Beatson [1]. He proved that the estimate

$$|f(x) - p_n(x)| \leq c \omega \left( f, \Delta_n(x) \right), x \in I, n = 1, 2, \ldots$$

$$\Delta_n(x) = \sqrt{\frac{1-x^2}{n} + \frac{1}{n^2}}$$

holds for suitable increasing polynomials $p_n$ whenever $f$ is increasing. Devore and Yn have shown that if $f$
is increasing function on \( I = [-1,1] \), then for each \( n = 1, 2, ... \) there is an increasing polynomial \( p_n \) of degree \( n \) such that

\[
|f(x) - p_n(x)| \leq c\omega_2 \left( f, \frac{1-x^2}{n} \right),
\]

where \( \omega_2 \) is the second order moduli of smoothness. Among other things, we shall show that this can be improved to allow second order modulus of smoothness for the spaces \( L_p, 0 < p < 1 \).

**Theorem I:** If \( f \) is an increasing function in \( L_p(I), 0 < p < 1 \) then for each \( n = 1, 2, ... \) there is an increasing polynomial in \( L_p(I) \) of degree \( 8n \) satisfying

\[
\|f - p_n\|_p \leq c(p)\omega_2^0 \left( f, \frac{1}{n} \right)_p. \tag{1.3}
\]

Using this theorem we can obtain our second Inverse inequality:

**Theorem II:** Let \( f \) be an increasing function in \( L_p(I), 0 < p < 1 \), then

\[
\omega_2^0 \left( f, n^{-1} \right)_p \leq c(p) E_n^1(f)_p + c(p)n^{-2p} \sum_{m>n} (m+1)^{p-1} E_n^1(f)_p.
\]
2. Auxiliary Lemmas

Before we prove our theorems we need the following notions and lemmas. Our proof is based on a two stage approximation. We first approximate $f$ by an increasing piecewise linear function $S_n$. We then approximate $S_n$ by an increasing algebraic polynomial. $S_n$ is the piecewise linear function that interpolates $f$ at $\xi_k, k = -n, \ldots, n$, if we let $s_j$ be the slopes

$$s_j = \frac{f(\xi_{j+1}) - f(\xi_j)}{\xi_{j+1} - \xi_j}, \quad j = -n, \ldots, n-1.$$  (2.1)

Then $S_n$ can be represented by using the function

$$\Phi_j(x) = \max\{|x - \xi_j, 0\}|$$
as

$$S_n(x) = f(-1) + s_{-n}(x + 1) + \sum_{j=n}^{n-1} (s_j - s_{j-1}) \Phi_j(x), \quad [4]$$  (2.2)

It is clear that $S_n$ is increasing if $f$ is also.

We shall now contract a polynomial $R_j, j = -n, \ldots, n-1$, as in [4] that approximate the function $\Phi_j(x)$. The construction of $R_j$ begins with trigonometric polynomial $T_j, j = 1, \ldots, 2n$ with
\[ t_j = \frac{j\pi}{2n}, \quad j = 0,1,\ldots,2n. \] Let \( K_n \) denote the Jackson kernel

\[
K_n(t) = a_n \left( \frac{\sin nt}{2} \right)^8 \sin \frac{t}{2},
\]

where \( a_n \) is a constant depending on \( n \) chosen so that

\[
\frac{\pi}{2n^2} \int_{-\pi}^{\pi} K_n(t) dt = 1.
\]

Here and throughout \( c(p), \ c, \) denote absolute constants depending on \( p \) and \( c(p), c \)’s, values may vary with each occurrence on the same line.

Define

\[
T_j(t) = \int_{t-t_j}^{t+t_j} K_n(u) du, \quad j = 0,1,\ldots,2n, [4]
\]

and define

\[
d_j(t) = \max(n \text{ dist } (t, [-t_j,t_j])), [4] \tag{2.4}
\]

Now let

\[
r_j(x) = T_{m-j}(t), \quad x = \cos t.
\]

And for \( x \in [-1,1] \) define

\[
R_j(x) = \int_{-1}^{x} r_j(u) du, \quad j = -n,\ldots,n, [4] \tag{2.5}
\]

In particular \( R_{-n}(x) = x + 1 = \Phi(x) \) and \( R_n(x) = 0 \), the points \( \xi_j \) are defined by the equations

\[ 1 - \xi_j = R_j(1), j = -n,\ldots,n. \] If \( f \in L_p(I) \) we define
\[ L_n(f) = f(-1) + s_{-n}R_{-n} + \sum_{j=n+1}^{n-1}(s_j - s_{j-1})R_j. \]  

with \( s_j \) defined by (2.1) if \( f \) is increasing, the \( s_j \geq 0, j = -n, \ldots, n-1 \) and since we can also write \( L_n(f) = f(-1) + \sum_{j=-n+1}^{n-1} s_j(R_j - R_{j+1}) \).

Now from the definition of the polynomials \( T_j \) we have \( T_{n-j} - T_{n-j+1} \geq 0 \), hence \( r_j - r_{j+1} \geq 0 \), and therefore \( R_j - R_{j+1} \) is increasing, it follows that \( L_n(f) \) is increasing.

We now estimate

\[ E(x) = S_n(x) - L_n(f, x) = \sum_{j=n+1}^{n-1}(s_j - s_{j-1})(\Phi_j(x) - R_j(x)). \]  

(2.7)

Now for \( j = -n, \ldots, n-1, x = \cos t \) with \( 0 \leq t \leq \pi \), we have

\[ |\Phi_j(x) - R_j(x)| \leq cn^{-1}(\sin t_{n-j} + |t - t_{n-j}|)(d_{n-j}(t))^{\frac{1}{p}}. \]  

(2.8)

Lemma 2.9. \( \|L_n(f)\|_p \leq c(p)\|f\|_p \)

Proof: We have

\[ S_n(x) - L_n(f, x) = \sum_{j=n+1}^{n-1}(s_j - s_{j-1})(\Phi_j(x) - R_j(x)). \]

Then

\[ |L_n(f, x)| = \left| S_n(x) - \sum_{j=n+1}^{n-1}(s_j - s_{j-1})(\Phi_j(x) - R_j(x)) \right| \]

\[ \leq |S_n(x)| + \left| \sum_{j=n+1}^{n-1}(s_j - s_{j-1})(\Phi_j(x) - R_j(x)) \right|. \]

Definition of \( S_n \) implies
\[ |S_n(x)| \leq |f(-1)| + |s_{-n}(x + 1)| + \sum_{j=-n+1}^{n-1} |s_j - s_{j-1}| \Phi_j(x). \]

Thus
\[ |S_n| \leq c|f(x)|. \]

And
\[ |L_n(f, x)| \leq c|f(x)| + \sum_{j=-n+1}^{n-1} |s_j - s_{j-1}| \Phi_j - R_j|. \]

Then (2.2.1) implies
\[ |s_j| \leq \frac{c|f(x)|}{\delta_j} \leq \frac{c|f(x)|}{\delta_{j+1}} \text{ where } \delta_j = \xi_{j+1} - \xi_j. \]

Then
\[ |L_n(f, x)| \leq c|f(x)| + c|f(x)| \sum_{j=-n+1}^{n-1} \delta_j^{-1} |\Phi_j - R_j|. \]

and by \( \delta_j = \xi_{j+1} - \xi_j = \frac{c}{n} \), we have \( \delta_j^{-1} \leq cn \).

So
\[ |L_n(f, x)| \leq c|f(x)| + c|f(x)| \sum_{j=-n+1}^{n-1} \left(1 + n|t - t_{n-j}|(d_{n-j}(t))^{-5} \right. \]
\[ \leq c|f(x)| + c|f(x)| \sum_{j=-n+1}^{n-1} \frac{1}{n^4}. \]

Then by the following [4]

**Lemma 2.10** If \( g' \) is absolutely continuous and \( |g''| \leq M \) almost everywhere on \( I \). Then for each \( n=1,2,... \) and each \( x \in I \), we have
\[ |g(x) - L_n(g, x)| \leq cM \left(\frac{\sqrt{1-x^2}}{n}\right)^2. \]
We can prove

**Lemma 2.11** If $g'$ is absolutely continuous and $|g'| \leq M$ almost every where on I, then for each $n = 1, 2, \ldots$, and each $x \in I$ we have

$$\|g - L_n(g)\|_p \leq \frac{c(p)}{n^2}.$$ 

**3. proof of theorem I**

Firstly let us introduce the so called Ditzian Totik functional definition as

$$\tilde{K}_{2, \varphi} \left( f, \frac{1}{n^2} \right) = \inf_{g} \left( \|f - g\|_p + \frac{1}{n^2} \|\varphi^* g^*\|_p \right),$$

for $f \in L_p(I), 0 < p \leq \infty$.

We have

$$\omega^*_p \left( f, n^{-1} \right) = \tilde{K}_{2, \varphi} \left( f, \frac{1}{n^2} \right) [2],$$

Given $x \in I$, then from the result above there is a $g$ satisfies

$$\|f - g\|_p \leq c(p) \omega^*_p \left( f, n^{-1} \right)$$

and

$$\frac{1}{n^2} \|\varphi^* g^*\| \leq c(p) \omega^*_p \left( f, n^{-1} \right)$$

(3.1)

$$\|f - L_n(f)\|_p \leq \|f - g\|_p + \|g - L_n(g)\|_p + \|L_n(g) - L_n(f)\|_p.$$ 

Then by the linearity, and the boundedness of $L_n(f)$, we obtain

$$\|f - L_n(f)\|_p \leq \|f - g\|_p + \|g - L_n(g)\|_p + \|L_n\|_p \|f - g\|_p$$

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\[ \leq \left( 1 + \| L_n \|_p \right) \| f - g \|_p + \| g - L_n(g) \|_p. \]

Lemma (2.11) implies \( \| g - L_n(g) \|_p \leq \frac{c(p)}{n^2} \).

Using (3.1), Lemma (2.11) and the linearity of \( L_n(f) \), we have

\[ \| f - L_n(f) \|_p \leq \left( 1 + \| L_n \|_p \right) c(p) \omega_2^\infty(f, n^{-1})_p^p + \frac{c(p)}{n^2} \| \varphi \|_p^p \]

\[ \leq \left( 1 + \| L_n \|_p \right) c(p) \omega_2^\infty(f, n^{-1})_p^p + c(p) \omega_2^\infty(f, n^{-1})_p^p. \]

By virtue of Lemma (2.9) we have

\[ \| f - L_n(f) \|_p \leq c(p) \left( 1 + \| f \|_p \right) \omega_2^\infty(f, n^{-1})_p^p + \omega_2^\infty(f, n^{-1})_p^p. \]

\[ \leq c(p) \omega_2^\infty(f, n^{-1})_p^p. \]

Since \( L_n(f) \) is an increasing polynomial of degree \( \leq 8n \) we have proved theorem 1 ♠

4. Proof of Theorem II

For \( \bar{x} \) given by \( = \max \{ i : 2^i < n \} \), \( = n \), we expand \( p_n(x) \) by

\[ p_n(x) - p_\bar{x}(x) = (p_n(x) - p_2(x)) + (p_2(x) - p_\bar{x}(x)) + \ldots + (p_\bar{x}(x) - p_\bar{x}(x)). \]

We recall that for \( m < n \)

\[ \| p_n - p_m \|_p \leq c(p) E_n^1(f)_p \]

\[ \omega_2^\infty(f, n^{-1})_p \leq c(p) \| f - p_n \|_p + c(p) n^{-p} \omega_2^\infty(p_n, n^{-1})_p^p \]

\[ \leq c(p) E_n^1(f)_p + c(p) n^{-2p} \| p_n' \|_p^p \]

\[ \leq c(p) E_n^1(f)_p + c(p) n^{-2p} \| \sum_i p_i' \|_p^p, \]

where \( p_\bar{x} \) is an algebraic polynomial of best monotone approximation of degree not greater than or equal to 2 it mean
\[ \| f - p^*_2 \|_p = E^1_2(f)_p. \] (4.1)

Then

\[ \omega_e^p(f, n^{-1})^p_p \leq c(p)E^1_n(f)_p^p + c(p)n^{-2p} \sum_{i=1}^{2^i} p^*_2 i - p^*_2 i-1 \|_p^p \]

\[ \leq c(p)E^1_n(f)_p^p + c(p)n^{-2p} \sum_{i=1}^{\infty} \| p^*_2 i - p^*_2 i-1 \|_p^p. \]

Then by Bernstein inequality we have

\[ \omega_e^p(f, n^{-1})^p_p \leq c(p)E^1_n(f)_p^p + c(p)n^{-2p} \sum_{i=1}^{\infty} (2^i n)^p E^1_{2^i n}(f)_p^p. \]

Applying the inequality

\[ 2^{np} \leq c(p, v) \sum_{m=2^{n+1}} (m+1)^{p-1}, v \in N \quad [5] \]

We get

\[ \omega_e^p(f, n^{-1})^p_p \leq c(p)E^1_n(f)_p^p + c(p)n^{-2p} \sum_{m>n} (m+1)^{p-1} E^1_m(f)_p^p. \]

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**References**


