

A Saturation theorem and negative theorem for monotone Approximation

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Abstract.

In [3], we introduce inverse and direct theorems for monotone approximation in $L_p, p < 1$, spaces. To complete the idea and characterize the monotone function in the generalized Lipschitz space through their approximation properties we introduce a theorem concerning the saturation of the increasing polynomial for this monotone approximation. One may hope that for a continuous increasing target function it would be possible to obtain estimate involving higher moduli of smoothness of this target function and this give better rate of monotone approximation. In our second result of this chapter we show that it is not possible to obtain direct theorem with higher moduli of smoothness when we assume the target function increasing with three continuous derivatives.

1 .Introduction and main results

A major problem of the theory of approximation of function is concerned with the connection between the structural properties of a function and its degree of the approximation. The objective is to relate the smoothness of the function to the rate of decrease of the degree of approximation to zero .We are interested in our paper in examining these questions for algebraic polynomial approximation . these are then the most classical settings where the results are the most penetrating and satisfying.

In many applications, it is desirable that the mathematical model preserves certain geometric properties of the data such as monotonicity .This is the subject that the so called monotone approximation.

Recently there have been more attention given to approximation with constrains. The appearance of constraints can make it more difficult to obtain direct estimates. still the general lines of attack developed for the non constrained problem can be very useful. We want to indicate what modifications are necessary push the approach through. We do this for monotone approximation.

In monotone approximation. We are given a monotone function f and we wish to approximate f by a monotone polynomials. The question then is: does the monotonic cost us any thing or can we achieve the same degree of approximation

as in the non constrained case ? The main purpose of this chapter is to provide answer to this question .

Interest in the subject began in the 1960's with work on monotone approximation by shishe [14] and by Lorentz and Zeller [11] , [12]. It gained momentum in the 197'0s and 1980's with work on monotone approximation of Devore, Beatson, Leviatan, Yu, Kopotun, Shevchuk and others[4],[5],[1],[6],[9],[10],[13], [8],and 1990's and 2000's with works on co approximation in [2].

Let us recall the Ditzan -Totik modulus of smoothness which defined for such an f as follows

$$\omega_r^\varphi(f, \delta, I) = \sup_{0 < h \leq \delta} \left\| \Delta_{h\varphi(\cdot)}^r(f, \cdot) \right\|_p, \quad \delta \geq 0 [2]$$

In the applications the φ usually used

$$\varphi(x) = (1 - x^2)^{1/2} \quad \text{for } x \in I = [-1, 1]$$

Where the r^{th} symmetric difference of f is given by

$$\Delta_h^r(f, x, [-1, 1]) = \Delta_h^r(f, x) = \begin{cases} \sum_{i=0}^r \binom{r}{i} (-1)^{r-i} f\left[x - \frac{rh}{2} + ih\right] & , x \pm \frac{rh}{2} \in [a, b] \\ 0 & o.w \end{cases}$$

In our work the approximation will be carried out by polynomial $p_n \in \Pi_n$, the space of polynomials of degree not exceeding n . Note that 1-monotone functions are just nondecreasing functions, respectively. we denote the class of all 1-monotone functions by Δ^1 .

Now for $f \in \Delta^1 \cap L_{p(I)}$, we denote by

$$E_n^1(f)_p = \inf_{p_n \in \Delta^1 \cap \Pi_n} \|f - p_n\|_p,$$

where the infimum is taken over all polynomials $p_n \in \Pi_n$.

Firstly in our negative theorem, we show that ω_2° can not be replaced by ω_3° , we may hope that for continuous function f , it would be possible to obtain estimates involving higher moduli of smoothness of f and this way have better rate of monotone approximation.

In this theorem we show that it is not possible to obtain a direct estimate with higher moduli of smoothness when we assume that

$f \in \Delta^1 \cap C^3(I)$. Namely, We prove :-

Theorem I: If $0 < \alpha < 2$, then an increasing function f is in Lip_α^* iff for each $n=1,2,\dots$, there is an increasing algebraic polynomial p_n of degree $8n$ such that

$$E_n^1(f)_p \leq \|f - p_n\|_p \leq c(p) \frac{1}{n^\alpha}.$$

Theorem II: For each n and $0 < p < 1$, there is a function $f \in \Delta^1 \cap C^3(I)$, such that for every polynomial $p_n \in \Delta^1 \cap \Pi_n$,

$$\text{either } \limsup_{x \rightarrow 1} \frac{\|f - p_n\|_p}{\varphi^3(x)} = \infty \quad (1.1)$$

$$\text{or } \limsup_{x \rightarrow 1} \frac{\|f - p_n\|_p}{\varphi^3(x)} = \infty, \quad x \in I.$$

Now with inverse and direct theorems for approximation by algebraic polynomials. We have the following result, which characterizes the lip_α^* spaces, which are defined as the set of all f such that

$$\omega_2^\varphi(f) = O(t^\alpha), \quad 0 < \alpha < 2.$$

2. Proof of theorem I

let f increasing function in lip_α^* from the direct theorem we have

$$\|f - p_n\|_p \leq \omega_2^\varphi\left(f, \frac{1}{n}\right)_p \leq c(p) \left(\frac{1}{n^\alpha}\right)$$

Inverse theorem implies

$$\begin{aligned} \omega_2^\varphi\left(f, n^{-1}\right)_p &\leq E_n^1(f)_p + c(p)n^{-2} \sum_{i=1}^n E_{i-1}(f)_p \\ &\leq c(p) \frac{1}{n^\alpha} + c(p)n^{-2} \sum_{i=2}^n \frac{1}{(i-1)^\alpha} \\ &\leq c(p) \frac{1}{n^\alpha} + c(p) \frac{1}{n^\alpha} \sum_{i=1}^{\infty} \frac{1}{i^2} \quad (\text{converging series}) \\ &\leq c(p) \frac{1}{n^\alpha} \quad \clubsuit \end{aligned}$$

3. Proof of theorem II

let $f(x) = \begin{cases} \frac{1}{3!} \left[\frac{1}{n^6} - \left[\frac{1}{n^2} - x - 1 \right]^3 \right], & -1 \leq x \leq -1 + \frac{1}{n^2} \\ \frac{1}{3!n^6}, & -1 + \frac{1}{n^2} \leq x \leq 1 \end{cases}$, f is non decreasing function in

$L_p [-1,1]$. In fact :

$$f'(x) = \begin{cases} \frac{\left(\frac{1}{n^2} - x - 1 \right)^2}{2}, & -1 \leq x \leq -1 + \frac{1}{n^2} \\ 0, & -1 + \frac{1}{n^2} \leq x \leq 1 \end{cases}$$

when $x = -1 + \frac{1}{n^4}$

$$f'\left(-1 + \frac{1}{n^4}\right) = \left(\frac{1}{n^2} - \left(-1 + \frac{1}{n^4}\right) - 1 \right)^2 / 2 = \left(\frac{1}{n^2} - \frac{1}{n^4} \right)^2 / 2 \geq 0.$$

suppose that there is a non decreasing polynomial p_n for which

(1.1) fails .

Then for that polynomial and some constant B , we have

$$\|f - p_n\|_p \leq B \|\varphi^3\|_p \quad \text{and} \quad \|f - p_n\|_\infty = \sup_{x \in [-1,1]} |f(x) - p_n(x)|$$

Then by using the inequality: $|p_n(x)| \leq c(p) n^{\frac{p}{2}} \|p_n\|_p$ [7]

$$|f(x) - p_n(x)| \leq c(p) n^{\frac{p}{2}} \|f - p_n\|_p \leq c(p) B n^{\frac{p}{2}} \|\varphi^3\|_p, \quad |f(-1) - p_n(-1)| \leq c(p) B n^{\frac{p}{2}} \|\varphi^3(-1)\|_p$$

$$\text{so } |f(-1) - p_n(-1)| = 0 \quad \text{and} \quad f(-1) = p_n(-1) = 0 \quad (3.1)$$

$$f(-1) = \frac{1}{3!} \left(\frac{1}{n^6} - \frac{1}{n^6} \right) = 0 \quad (3.2)$$

and

$$f'(x) = \left(n^{-2} - x - 1 \right)^2 / 2, \quad -1 \leq x \leq -1 + \frac{1}{n^2}$$

$$f'(-1) = \frac{n^{-4}}{2}$$

thus by (3.2) we have

$$f'(-1) = p'_n(-1) = \frac{n^{-4}}{2}.$$

Then applying the inequality $|p'_n| \leq c(p) n^{2+\frac{2}{p}} \|p_n\|_p$, [17]

$$f(1) = \frac{1}{3!n^6} \leq \frac{n^{-4}}{2} = p'_n(-1) \leq c(p) n^{2+\frac{2}{p}} \|p_n\|_p \leq n^4 \|p_n\|_p$$

$$\text{Now } n^4 \|p_n\|_p = n^4 \left(\int_{-1}^1 |p_n|^p \right)^{\frac{1}{p}}$$

$$\leq n^4 \left(2 |p_n(1)|^p \right)^{\frac{1}{p}}$$

$$= n^4 2^{\frac{1}{p}} p_n(1)$$

$$\leq n^5 p_n(1).$$

And $n^{-6}/3! \leq n^{-4}/2 \leq n^5 p_n(1)$, and $f(1) = n^{-6}/3! \leq n^5 p_n(1)$.

Thus $f(1) \neq p_n(1)$ and (1.1) satisfied ♣

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