The Local Connectivity Of The Mandelbrot Sets Of $M(Q_\lambda)$

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Abstract :

In this paper , we show that the Mandelbrot sets of the quadratic polynomial of the form $Q_\lambda(z) = \lambda z - \lambda z^2$, Where λ is a non-zero complex constant parameter, is locally connected at λ .

المستخلص :

في هذا البحث ، سوف نبين أن المجموعة المندلبروت للدالة التربيعية ذات الصيغة ($\lambda z - \lambda z^2$)، حيث أن λ هو عدد غير صفري ، هي متصلة محليا .

1- Introduction

The local theory of iterated analytic mappings developed in the late 19 th century due to Schroder , Poincare ,Leau, Koenigs and Bottcher [10] . Douady and Hubbard proved that M (M is Mandelbrot set) is connected [8] . A problem is to study the local connectivity of the Julia set $J(g_c)$ when $c \in M$,where $g_c(z) = z^2 + c$, c is a complex parameter .Also, Douady and Hubbard gave an example of an infinitely renormalizable g_c whose Julia set is connected but not locally connected [7] . For an application of these general Puzzles to the study of the local connectivity of the Mandelbrot set , see [3] . The next important problem in this direction is to show that the Mandelbrot set is locally connected. By [9] , $Q_{\lambda}(z) = \lambda z - \lambda z^2$ conjugate to g_c ,where $\lambda = 1 \pm \sqrt{1-4c}$.

2- Preliminaries

The goal of this section is to give the basic definitions and the concepts for this paper .

Definition (2.1) [9]

Let $f : X \to X$ be the map .Suppose x is a fixed point of f.Then the basin of attraction of x consists of all y such that $f^n(y)$ converges to x as $n \to \infty$.

Definition (2.2) [9]

Suppose $f : C \to C$ is an analytic map .The Julia set is the closure of all repelling periodic points of f .That is

$$J(f) = \text{ closure \{ all repelling periodic points of } f \text{ \}}$$

We put $C_\infty = C \cup \{\infty\}$.

Definition (2.3) [9]

Suppose $f : C_\infty \to C_\infty$ is an analytic map . A point z in C is called a super attracting fixed point if $|f'(z)| = 0$.

Definition (2.4) [9]

Suppose $f : C_\infty \to C_\infty$ is an analytic map .

$K(f)$ = { $z \in C : \{f^n(z)\}_{n=0}^{\infty}$ is bounded } .That set is called

Filled Julia set .

Definition (2.5) [6]

A topological space X is local connected $x \in X$ if there exist arbitrarily small connected neighborhoods of x in X .

Lemma (2.6) [6]

The space X is locally connected at every point $x \in X$ iff every open subset of X is a union of connected open subsets of X.

Proof: see [6].

Definition (2.7) [1] [pp.315]

The Mandelbrot set is the subset of the c-plane given by

$$M = \{c : g_c^n(0) \not\to \infty\}$$.

Since 0.5 is the critical point of Q_λ then we recall :

$$M = \{\lambda : Q_\lambda^n(0.5) \not\to \infty\}$$.

Definition (2.8) [7]

A map g_c is hyperbolic (on its Julia set) iff it has a necessarily unique attracting periodic orbit .

Theorem (2.9) [5]

The map g_c is hyperbolic iff $g_c^n(0) \to \infty$,or g_c has an attracting periodic cycle in the plane C .

Definition (2.10) [2]

Let $f : C \to C$ be a polynomial of degree n and let $U = D_r$ be the disc of radius r .If r is large enough then $U' = f^{-1}(U)$ is relatively compact in U and homeomorphic to a disc, and $f : U' \to U$ is analytic, proper of degree n.

Definition (2.11) [2]

A polynomial-like map of degree n is a triple (U, U', f) where U and U' are open subsets of C isomorphic to disc ,with U' relatively compact in U ,and $f : U' \to U$ a C - analytic map ,proper of degree n .

Definition (2.12) [5]

An iterate f^n is renormalizable if there exist disks U and V containing the origin, with $\overline{U}$ a compact subset of V , such that (a) $f^n : U \to V$ is a proper map of degree two .

(b) $f^{nk}(0) \in U$ for all $k > 0$,where 0 is critical point .

Definition (2.13) [5]

A quadratic polynomial f is infinitely renormalizable if f^n is renormalizable for infinitely many $n > 1$.

Definition (2.14) [9]

Let $f : C_\infty \to C_\infty$ be an analytic map has a super attracting fixed point at 0 .Let Ω be the basin of attraction of this fixed point. Define $G : \Omega - \{0\} \to \Re$ ($\Re$ real number) by $G(z) = \log|\Phi(z)|$ .If $G(z) < 0$ .Then the map is called the Green `s maps of f .

3- The local connectivity of $M(Q_\lambda)$

We will study in this section the locally connected of the Mandelbrot set for the map Q_λ .

Remark (3.1) [6]

Let $\overline{D}$ be the closed disk and K is filled Julia set ,the complement $C \setminus K$ is isomorphic to the complement $C \setminus \overline{D}$ under a conformal isomorphism $\Phi : C \setminus K \longrightarrow C \setminus \overline{D}$ such that $\Phi(f(z) = \Phi(z)^d$. (Φ the Riemann map) .

Definition (3.2) [6]

Let $\Psi : C \setminus \overline{D} \longrightarrow C \setminus K$ be the Riemann inverse map . The image under Ψ of a radial line { $r e^{2\pi i t}$: $r > 1$ } in $C \setminus \overline{D}$ is called the external ray R_t at angle t in $C \setminus K$. Note that our angles are element of $\Re / Z$ ($\Re$ real number, Z integer number) , measured in fractions of a full turn and not in radians . $f(R_t) = R_{dt}$.

Remark (3.3) [7]

If a connected quadratic Julia set has two repelling fixed points , then one fixed point (to be called β) is the landing point of the zero external ray , and the other (called α) is the landing point of a cycle of q external rays, where $q \geq 2$ .Let $0 = c_0 \rightarrow c_1 \rightarrow \dots$ be the critical orbit. The Yoccoz Puzzle of depth zero consists of q non-overlapping closed topological disks $P_0(c_0), P_0(c_1), \dots, P_0(c_{q-1})$,called "Puzzle pieces" .

Lemma (3.4) [4]

Let diam (U_n) denote the diameter of U_n .Then

$$\text{diam } (U_n) \to 0 \quad \text{as} \quad n \to \infty \ .$$

Proof : see as [4] .

Definition (3.5) [5]

An invariant line field for f , defined on a measurable set $E \subset C$, is the choice of a 1-dimensional real subspace L_z in the tangent space $T_z C$ for all $z \in E$,such that :

1. E has positive area ;
2. $f^{-1}(E) = E$;
3. the slope of L_z varies measurably with respect to z ; and
4. the derivative f' transforms L_z into $L_{f(z)}$ for all z in E .

Remark (3.6) [5]

If $E \subset J(f)$ we say f admits an invariant line field on its Julia set .

Theorem (3.7) [5]

If the Mandelbrot set is locally connected , then the hyperbolic maps are dense and Julia set no invariant line field .

Proof : see as [5] .

Proposition (3.8) [Conjecture to Yoccoz theorem]

If λ belongs to the Mandelbrot set , then either : $Q_\lambda(z) = \lambda z - \lambda z^2$ is infinitely renormalizable or $J(Q_\lambda)$ admits no invariant line field and M is locally connected at λ .

We will use their language of tableaux (the tableau associated with an orbit $z_0 \mapsto z_1 \mapsto z_2 \mapsto . . .$ in Julia set is an array with one column associated with each z_i and one row associated with each depth in the Yoccoz puzzle) to describe proof .

Proof :

Suppose $\lambda \in M$ .Let $K(Q_\lambda)$ is filled Julia set which remain bounded under iteration Q_λ ; its boundary is the Julia set and $K(Q_\lambda)$ is connected . Let $\phi_\lambda : (C - \overline{\Delta}) \to (C - K(Q_\lambda)$ be the Riemann map , normalized so $\phi'_\lambda(z) = 1$ at infinity . It is $\phi_\lambda(z^2) = Q_\lambda(\phi_\lambda(z))$;in other words ϕ_λ conjugates the z^2 to Q_λ . An external ray R_t is the image of the ray $(1, \infty) \exp(2\pi it)$ under the map Q_λ ; similar , an external circle λ_r (also called an equipotential) is the image of $\{z : |z| = r\}$ .Note that $Q_\lambda(R_t) = R_{2t}$ and $Q_\lambda(\lambda_r) = \lambda_{r^2}$ by the functional equation for ϕ_λ . Then main case of the proof arises when all periodic cycles of Q_λ are repelling ; let us assume this . The first step is to try show that the Julia set $J(Q_\lambda)$ is locally connected .To this end , Yoccoz constructs a sequence $\langle P_n \rangle$ of successively finer tilings of neighborhoods of $J(Q_\lambda)$ .To illustrate the method, consider the special case $\lambda = 1 \pm \sqrt{1 - 4i}$ .For this map , the external rays $R_{\frac{1}{7}}$, $R_{\frac{2}{7}}$ and $R_{\frac{4}{7}}$ converge to a repelling fixed point α' of Q_λ .These rays cut the disk bounded by the external circle λ_2 into three tiles (see Figure (1)) , called the puzzle pieces ρ_0 at level 0 . The pieces at level (n+1) are defined inductively as the components of the preimages of the pieces ρ_n at level n . The new pieces fit neatly inside those already defined ,because the external rays converging to α' are forward –invariant . The puzzle pieces provide connected neighborhoods of points in the Julia set .To show $J(Q_\lambda)$ is locally connected , it suffices to show that $diam(\mathrm{P}_n) \to 0$ for any nested sequence of pieces $\mathrm{P}_0 \supset \mathrm{P}_1 \supset \mathrm{P}_2$ … . Now $diam(\mathrm{P}_i) \to 0$ will follow if we can establish $\sum \mathrm{mod}(\mathrm{P}_i - \mathrm{P}_{i+1}) = \infty$; here each region $\mathrm{P}_i - \mathrm{P}_{i+1}$ is a (possibly degenerate) annulus , and the modulus mod(A) = m if the annulus A is conformally isomorphic to the standard round annulus { z : 1 < $|z|$ < exp (2π m) } .The modulus is especially useful in holomorphic dynamics because it is invariant under conformal maps ; more generally ,mod (A`) = mod (A)/ n if A` is a n-fold covering of an annulus A . Since the image of a puzzle piece of depth n > 0 under Q_λ is again a puzzle piece , the moduli of the various annuli that can be formed satisfy many relations .Roughly speaking , the tableau method allows one to organize these relations and test for divergence of sums of moduli. For degree two polynomials , the method succeeds unless certain annuli are repeatedly covered by degree two .Unfortunately , this exceptional case leads to the convergent sum 1+1/2+1/4+ **…** and so it does not prove local connectivity . However one finds this case only occurs when the polynomial is renormalizable . The case of a finitely renormalizable map Q_λ can be handled by respecifying the initial tiling ρ_0 . Thus the method establishes locally connectivity of $J(Q_\lambda)$ unless the map is infinitely renormalizable .It is a metatheorem that the structure of the Mandelbrot set at λ reflects properties of the Julia set $J(Q_\lambda)$ .In this the case the proof of locally connectivity of $J(Q_\lambda)$ can be adapted , with some difficulty ,to establish locally connectivity of M at λ . By Theorem (3.7) then shows $J(Q_\lambda)$ admits no invariant line field . □

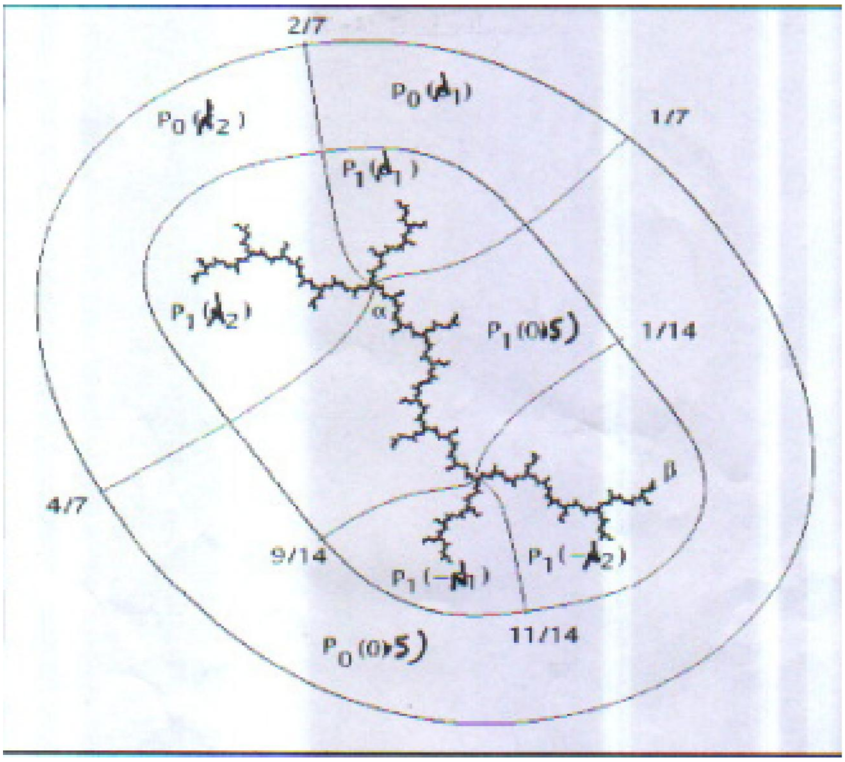

Figure (1)

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