The Non-Central *t*-Distribution and How Finding Asymptotic Expansion for this Distribution

By
Jinan Hamzah Farhood
Department of Mathematics
College of Education
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Abstract

We introduce in this paper the incomplete beta function , the non-central *t*-distribution and derive an asymptotic expansion of this distribution ,where t' is a variable has a non-central *t*-distribution with v_2 degrees of freedom (positive integer)and non-central parameter δ (real), $Q\left(\frac{v_2}{2},-z\log q\right)=\frac{\Gamma\left(\frac{v_2}{2},-z\log q\right)}{\Gamma\left(\frac{v_2}{2}\right)}$ is the incomplete Gamma function ratio , $q=\frac{\frac{v_1F'}{v_2}}{1+\frac{v_1F'}{v_2}}$, F' is said to have a non-central *F*-distribution with v_1 ,v_2 degrees of freedom (positive integer) , $Z=\left(\frac{r+1}{2}\right)+\frac{\left(\frac{v_2}{2}-1\right)}{2}$, S_1 and S_2 are signs differing between cases with positive or negative *t* as well as odd or even *r* in the summation .

توزيع *t* اللامركزي وكيفية إيجاد الصيغة التقاربية لهذا التوزيع

من قبل
جنان حمزة فرهود
قسم الرياضيات –كلية التربية
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الخلاصة

نقدم في هذا البحث دالة بيتا الغير كاملة وتوزيع *t* اللامركزي ونقوم باشتقاق الصيغة التقاربية لهذا التوزيع حيث إن t' يمثل متغير يمتلك توزيع*t* اللامركزي بدرجة حرية v_2 والتي تكون عدد صحيح موجب وبمعلمة لامركزية δ والتي تكون عدد حقيقي بينما $Q\left(\frac{v_2}{2},-z\log q\right)=\frac{\Gamma\left(\frac{v_2}{2},-z\log q\right)}{\Gamma\left(\frac{v_2}{2}\right)}$ تمثل نسبة دالة كاما

الغير كاملة بحيث إن $q = \frac{\frac{v_1 F'}{v_2}}{1+\frac{v_1 F'}{v_2}}$ ،F' يمتلك توزيع F اللامركـــزي بـــدرجتي حريـــة v_1 ،v_2 وكـــذلك

أما $Z = \left(\frac{r+1}{2}\right) + \frac{\left(\frac{v_2}{2}-1\right)}{2}$ S_1 و S_2 فتمثلان إشارات مختلفة بين حالتين والتي تكون فيها t موجبة أو سالبة ويمثل r عدد زوجي أو فردي في المجموع .

1 Introduction

The non-central t-distribution $F(t'\backslash v_2, \delta)$ is defined by (Henry, 1959; Walkk, 2001) and it is studied by other researchers .The handbook of mathematical functions introduced by (Abramowitz & Stegun ,1970) whose defined the incomplete beta function while the computation of the incomplete gamma function ratios and their inverse studied by (Didonato & Morris ,1986) .

Owing to the wide variation in behavior in different regions of the parameter space , efficient code to evaluate $I_q(\frac{r+1}{2}, \frac{v_2}{2})$ involves a number of different subroutines for different parts of this parameter space .In this paper we shall confine our interest to a subdomain of the parameter space in which $\frac{r+1}{2}$ is large , $\frac{v_2}{2}$ is small and q is close to 1 . Indeed if $\frac{v_2}{2}$ <1 , then $I_q(\frac{r+1}{2}, \frac{v_2}{2})$ varies most rapidly as q approaches 1 . This region has to be treated very carefully . Asymptotic expansions suitable for this subdomain have been derived by (Temme,1987) . These asymptotic expansions have the form $\Sigma A_n/Z^n$,where Z is either $\frac{r+1}{2}$ or $(\frac{r+1}{2}) + \frac{(\frac{v_2}{2}-1)}{2}$ (Didonato & Morris ,1992) , and in which the expansion coefficients A_n depend on all three parameters $\frac{r+1}{2}, \frac{v_2}{2}$ and q . The expansion to be described here has the same general form ,but the expansion coefficients A_n depend only on $\frac{v_2}{2}$ and q . The advantage of this new expansion is that it is cleaner and that an algorithm based on it can be more easily tuned for particular accuracy requirements and for particular parameter ranges .

We introduce the asymptotic and asymptotic expansion for different subjects which used by many mathematicians as the form .

On the asymptotics of the jacobi function and its zeros studied by (Wong & Wang , 1992) while asymptotic and numerical aspects of the non central chi-square distribution derived by(Temme,1993),symbolic integration and asymptotic expansions studied by (Cohen,1995) , uniform asymptotic for the incomplete gamma functions starting from negative values of the parameters derived by (Temme,1996) and numerical Algorithms for uniform Airy- type asymptotic expansions introduced by (Temme,1997) , in the same year asymptotic expansions of the generalized Bessel polynomials derived by (Wong & Zhang,1997) and the asymptotics of a second solution to the jacobi differential equation is also derived by (Wong & Zhang,1997) .

On the high-order coefficients in the uniform asymptotic expansion for the incomplete gamma function introduced by (Dunster,Paris & Cang ,1998) but asymptotic approximations for the jacobi and ultraspherical polynomials, and related functions, methods and applications of analysis studied by (Dunster,1999) while a uniform asymptotic expansion for krawtchouk polynomials derived by (Li & Wong,2000) .

Asymptotic expansions in non-central limit theorems for Quadratic forms derived by (Gotze & Tikhamirov,2001), in the same year asymptotics and mellin-barnes integrals studied by (Paris & Kaminski,2001) while an asymptotic expansions for a ratio of products of gamma functions derived by (Wolfgang,2003) and an introduction in asymptotic analysis introduced by (Simon,2004) .

2 Definition of the non-Central *t*-Distribution (Henry,1959;walk, 2001).

If X and Y are independent random variables and X is a normal distribution with mean δ and variance 1 and Y is a central chi-square with v_2 degrees of freedom then the variable $t' = \dfrac{X}{\sqrt{Y/v_2}}$ has a non-central *t*-distribution with v_2 degrees of freedom (positive integer) and non-central parameter δ (real) and we write $t' \sim t_{v_2,\delta}$. The distribution function is given by

$$F(t'\backslash v_2,\delta)=\frac{e^{-\delta^{2/2}}}{\sqrt{\pi}}\sum_{r=0}^{\infty}\frac{\delta^2}{r!}\,2^{\frac{r}{2}-1}\,\Gamma\Big(\frac{r+1}{2}\Big)\left\{S_1+S_2 I_q\Big(\frac{r+1}{2},\frac{v_2}{2}\Big)\right\}, \tag{1}$$

Which is the cumulative distribution function (c.d.f.)of non-central *t*-distribution, while best known for its applications in statistics, it is also widely used in many other fields .Where S_1 and S_2 are signs differing between cases with positive or negative t as well as odd or even r in the summation .The sign S_1 is -1 if r is odd and +1 if it is even while S_2 is +1 unless $t<0$ and r is even in which case it is -1 ,and

$$I_q\Big(\frac{r+1}{2},\frac{v_2}{2}\Big)=\frac{B_q\Big(\frac{r+1}{2},\frac{v_2}{2}\Big)}{B\Big(\frac{r+1}{2},\frac{v_2}{2}\Big)}=\frac{\Gamma\Big(\frac{r+1}{2}+\frac{v_2}{2}\Big)}{\Gamma\Big(\frac{r+1}{2}\Big)\Gamma\Big(\frac{v_2}{2}\Big)}\int_0^q u^{\frac{r+1}{2}-1}\left(1-u\right)^{\frac{v_2}{2}-1}du, \tag{2}$$

this equation is called the incomplete beta function such that $q=\dfrac{\frac{v_1F'}{v_2}}{1+\frac{v_1F'}{v_2}}$ and

$F'=\dfrac{X/v_1}{Y/v_2}$ is said to have a non-central *F*-distribution with v_1, v_2 degrees of freedom (positive integers).

3 Derivation of an Asymptotic Expansions for the non-Central *t*-Distribution .

We derive an asymptotic expansion of $F(t'\backslash v_2,\delta)$, through two sections .

3.1 Asymtotic Expansion of $\Gamma(\frac{r+1}{2})/\Gamma(\frac{r+1}{2}+\frac{v_2}{2})$

In this section we shall derive an asymptotic expansion which we shall need later , which provides an efficient method of calculating $\Gamma(\frac{r+1}{2})/\Gamma(\frac{r+1}{2}+\frac{v_2}{2})$ when $\frac{r+1}{2}\geq\frac{v_2}{2}$ and $r\in N$. We start from the beta function $B(\frac{r+1}{2},\frac{v_2}{2})$.

$$B\left(\frac{r+1}{2},\frac{v_2}{2}\right)=\frac{\Gamma(\frac{r+1}{2})\Gamma(\frac{v_2}{2})}{\Gamma(\frac{r+1}{2}+\frac{v_2}{2})}=\int_0^1 u^{\frac{r+1}{2}-1}\left(1-u\right)^{\frac{v_2}{2}-1}du \quad . \tag{3}$$

Then by using the substitution $u = e^{-w}$ *and* $du = - e^{-w}\,dw$ we obtain

$$B\left(\frac{r+1}{2},\frac{v_2}{2}\right)=\int_0^\infty e^{-\left(\frac{r+1}{2}\right)w}\left(1-e^{-w}\right)^{\frac{v_2}{2}-1}dw. \tag{4}$$

And using the fact that $(1- e^{-w}) = e^{-w/2}\, 2\sinh\frac{w}{2}$ we have

$$B\left(\frac{r+1}{2},\frac{v_2}{2}\right)=\int_0^\infty e^{-zw}w^{\frac{v_2}{2}-1}\left[\sinh\left(\frac{w}{2}\right)/\left(\frac{w}{2}\right)\right]^{\frac{v_2}{2}-1}dw, \tag{5}$$

where $z=(\frac{r+1}{2})+\frac{(\frac{v_2}{2}-1)}{2}$. We now expand $\left[\sinh\left(\frac{w}{2}\right)/\left(\frac{w}{2}\right)\right]^{\frac{v_2}{2}-1}$ in powers of w^2 as $w\to\infty$.

$$\left[\sinh\left(\frac{w}{2}\right)/\left(\frac{w}{2}\right)\right]^{\frac{v_2}{2}-1}= \left[\sum_{n=0}^{\infty}\frac{w^{2n}}{(2n+1)!2^{2n}}\right]^{\frac{v_2}{2}-1}=\left[\sum_{n=0}^{\infty}h_n w^{2n}\right]^{\frac{v_2}{2}-1}\sim\sum_{n=0}^{\infty}C_n w^{2n}. \tag{6}$$

Let $h_n=\left[\sum_{n=0}^{\infty}\frac{1}{(2n+1)!2^{2n}}\right]^{\frac{v_2}{2}-1}$ .The last quality follows by (Didonato & Morris ,1992) . Where C_n are the expansion coefficients of $\left[\sinh\left(\frac{w}{2}\right)/\left(\frac{w}{2}\right)\right]^{\frac{v_2}{2}-1}$ .The coefficients C_n can be expressed in terms of the generalized Bernoulli polynomials (Luke,1969), $C_n = B_{2n}^{1-\frac{v_2}{2}}\frac{\left(\frac{1-\frac{v_2}{2}}{2}\right)}{(2n)!}$. By substitution equation (6) in (5) we get

$$B\left(\frac{r+1}{2},\frac{v_2}{2}\right)\sim\int_0^\infty e^{-zw}w^{\frac{v_2}{2}-1}\sum_{n=0}^{\infty}C_n w^{2n}dw$$

$$B\left(\frac{r+1}{2},\frac{v_2}{2}\right)\sim\sum_{n=0}^{\infty}C_n\int_0^\infty e^{-zw}w^{\frac{v_2}{2}+2n-1}dw.$$

And using Watson's Lemma we obtain the asymptotic expansion

$$\frac{\Gamma\left(\frac{r+1}{2}\right)}{\Gamma\left(\frac{r+1}{2}+\frac{v_2}{2}\right)} \sim \frac{1}{z^{\frac{v_2}{2}}} \sum_{n=0}^{\infty} C_n \frac{\Gamma\left(\frac{v_2}{2}+2n\right)}{\Gamma\left(\frac{v_2}{2}\right)} \left(\frac{1}{z}\right)^{2n} \tag{7}$$

3.2 Asymptotic Expansion of the Incomplete Beta Function $I_q\left(\frac{r+1}{2}, \frac{v_2}{2}\right)$.

In this section we transform the expression for $I_q\left(\frac{r+1}{2}, \frac{v_2}{2}\right)$ in (2) in the same way as in equation (3) to obtain

$$I_q\left(\frac{r+1}{2}, \frac{v_2}{2}\right) = \frac{\Gamma\left(\frac{r+1}{2}+\frac{v_2}{2}\right)}{\Gamma\left(\frac{r+1}{2}\right)\Gamma\left(\frac{v_2}{2}\right)} \int_{-\log q}^{\infty} e^{-zw} w^{\frac{v_2}{2}-1} \left[\sinh\left(\frac{w}{2}\right)/\left(\frac{w}{2}\right)\right]^{\frac{v_2}{2}-1} dw, \tag{8}$$

where as before $z = \left(\frac{r+1}{2}\right) + \frac{\left(\frac{v_2}{2}-1\right)}{2}$, by using (6) we have

$$I_q\left(\frac{r+1}{2}, \frac{v_2}{2}\right) \sim \frac{\Gamma\left(\frac{r+1}{2}+\frac{v_2}{2}\right)}{\Gamma\left(\frac{r+1}{2}\right)\Gamma\left(\frac{v_2}{2}\right)} \sum_{n=0}^{\infty} C_n \int_{-\log q}^{\infty} e^{-zw} w^{2n+\frac{v_2}{2}-1} dw. \tag{9}$$

Let $f = zw$,then $w = \frac{1}{z} f$ and $dw = \frac{1}{z} df$, so from (9) we have

$$I_q\left(\frac{r+1}{2}, \frac{v_2}{2}\right) \sim \frac{\Gamma\left(\frac{r+1}{2}+\frac{v_2}{2}\right)}{\Gamma\left(\frac{r+1}{2}\right)\Gamma\left(\frac{v_2}{2}\right)} \sum_{n=0}^{\infty} C_n \int_{-z\log q}^{\infty} e^{-f} \left(\frac{1}{z} f\right)^{2n+\frac{v_2}{2}-1} \frac{1}{z} df$$

$$= \frac{\Gamma\left(\frac{r+1}{2}+\frac{v_2}{2}\right)}{\Gamma\left(\frac{r+1}{2}\right)\Gamma\left(\frac{v_2}{2}\right)} \sum_{n=0}^{\infty} C_n \frac{\Gamma\left(\frac{v_2}{2}+2n\right)}{z^{2n+\frac{v_2}{2}} \Gamma\left(\frac{v_2}{2}+2n\right)} \int_{-z\log q}^{\infty} e^{-f} f^{2n+\frac{v_2}{2}-1} df$$

$$\sim \frac{\Gamma\left(\frac{r+1}{2}+\frac{v_2}{2}\right)}{\Gamma\left(\frac{r+1}{2}\right)\Gamma\left(\frac{v_2}{2}\right) z^{\frac{v_2}{2}}} \sum_{n=0}^{\infty} \frac{C_n}{z^{2n}} \Gamma\left(\frac{v_2}{2}+2n\right) Q\left(\frac{v_2}{2}+2n, -z\log q\right), \tag{10}$$

where *Q(.,.)* is incomplete gamma function ratio . We can proceed by using the recurrence relations for *Q(.,.)* to express $Q\left(\frac{v_2}{2}+2n, -z\log q\right)$ in terms of $Q\left(\frac{v_2}{2}, -z\log q\right)$. This gives

$$I_q\left(\frac{r+1}{2}, \frac{v_2}{2}\right) \sim Q\left(\frac{v_2}{2}, -z\log q\right) + R\left(\frac{r+1}{2}, \frac{v_2}{2}, q\right), \tag{11}$$

where we have use the formula (7) to cancel out the factors multiplying Q ,the other term $R\left(\frac{r+1}{2},\frac{v_2}{2},q\right)$ is a double summation over n and $2n$ residual terms obtained by expressing $Q\left(\frac{v_2}{2}+2n,-z\log q\right)$ in terms of $Q\left(\frac{v_2}{2},-z\log q\right)$.

We can obtain the asymptotic expansion we require to reordering this sum. First we write (8) in the form

$$I_q\left(\frac{r+1}{2},\frac{v_2}{2}\right)=\frac{\Gamma\left(\frac{r+1}{2}+\frac{v_2}{2}\right)}{\Gamma\left(\frac{r+1}{2}\right)\Gamma\left(\frac{v_2}{2}\right)}\Big[\int_{-\log q}^{\infty}e^{-zw}\left(\left[2\sinh(w/2)\right]^{\frac{v_2}{2}-1}-w^{\frac{v_2}{2}-1}\right)dw$$
$$+\int_{-\log q}^{\infty}e^{-zw}w^{\frac{v_2}{2}-1}dw\Big] \tag{12}$$

Integrate the first integral by parts twice as follows

$$\int_{-\log q}^{\infty}e^{-zw}\left(\left[2\sinh(w/2)\right]^{\frac{v_2}{2}-1}-w^{\frac{v_2}{2}-1}\right)dw$$
$$=\frac{1}{z^2}\int_{-\log q}^{\infty}e^{-zw}\frac{d^2}{dw^2}\left(\left[2\sinh(w/2)\right]^{\frac{v_2}{2}-1}-w^{\frac{v_2}{2}-1}\right)dw$$
$$+\frac{q^z}{z}\left[\left[2\sinh(w/2)\right]^{\frac{v_2}{2}-1}-w^{\frac{v_2}{2}-1}+\frac{1}{z}\frac{d}{dw}\left(\left[\left[2\sinh(w/2)\right]^{\frac{v_2}{2}-1}-w^{\frac{v_2}{2}-1}\right]\right)\right]\Big|_{w=-\log q} \tag{13}$$

In the integral in (13)we now subtract the second term $\left(C_1w^{\frac{v_2}{2}+1}\right)$ in expansion of $\left[2\sinh(w/2)\right]^{\frac{v_2}{2}-1}$ and add a corresponding integral so that the integral in (13) becomes

$$\int_{-\log q}^{\infty}e^{-zw}\frac{d^2}{dw^2}\left(\left[2\sinh(w/2)\right]^{\frac{v_2}{2}-1}-w^{\frac{v_2}{2}-1}-C_1w^{\frac{v_2}{2}+1}\right)dw+\frac{\Gamma\left(\frac{v_2}{2}+2\right)}{\Gamma\left(\frac{v_2}{2}\right)}\int_{-\log q}^{\infty}e^{-zw}w^{\frac{v_2}{2}-1}dw, \tag{14}$$

and producing two further integrated terms evaluated at *w = -logq* .

Again integral the integral term in (13) by parts twices and subtract a third term $\left(C_2w^{\frac{v_2}{2}+3}\right)$ from the expansion of $\left[2\sinh(w/2)\right]^{\frac{v_2}{2}-1}$ and add a corresponding integral on separately . This procedure is continued indefinitely . The separate integrals stating from the ones on the right of (12) and (14) add together to give $Q\left(\frac{v_2}{2},-z\log q\right)$ as in (11) so that

$$I_q\left(\frac{r+1}{2},\frac{v_2}{2}\right)\sim Q\left(\frac{v_2}{2},-z\log q\right)+\frac{\Gamma\left(\frac{r+1}{2}+\frac{v_2}{2}\right)}{\Gamma\left(\frac{r+1}{2}\right)\Gamma\left(\frac{v_2}{2}\right)}q^z\sum_{n=0}^{\infty}\frac{T_n\left(\frac{v_2}{2},q\right)}{z^{n+1}}, \tag{15}$$

where

$$T_n\left(\frac{v_2}{2},q\right)=\frac{d^n}{dw^n}\left(\left[2\sinh(w/2)\right]^{\frac{v_2}{2}-1}-\sum_{m=0}^{n/2}C_m w^{2m+\frac{v_2}{2}-1}\right)\Big|_{w=-\log q}=\frac{d^n}{dw^n}\left[\sum_{m=\frac{n}{2}+1}^{\infty}C_m w^{2m+\frac{v_2}{2}-1}\right]\Big|_{w=-\log q}$$

and *n/2* in the summation is to be interpreted as largest integer $\le n/2$ as in integer division . The quantities T_n satisfy the simple recurrence formula $T_{2n+1}=\frac{d}{dw}T_{2n}$,

$$T_{2n}=\frac{d}{dw}T_{2n-1}-C_n w^{\frac{v_2}{2}-1}\frac{\Gamma\left(2n+\frac{v_2}{2}\right)}{\Gamma\left(\frac{v_2}{2}\right)} \quad . \tag{16}$$

We can express $T_n\left(\frac{v_2}{2},q\right)$ directly in terms of $\frac{v_2}{2}$ and q , for example ,

$$T_0\left(\frac{v_2}{2},q\right)=\left(1/\sqrt{q}-\sqrt{q}\right)^{\frac{v_2}{2}-1}-\left(-\log q\right)^{\frac{v_2}{2}-1} .$$

However, for q close to 1 , evaluation of T_n in this way can lead to large rounding errors on subtraction , and so $T_n\left(\frac{v_2}{2},q\right)$ is better evaluated from its power series expansion in w . Now , when we substitute the formula (15) in equation (1) ,we get

$$F(t'\backslash v_2,\delta)\sim\frac{e^{-\frac{\delta^2}{2}}}{\sqrt{\pi}}\sum_{r=0}^{\infty}\frac{\delta^2}{r!}2^{\frac{r}{2}-1}\Gamma\left(\frac{r+1}{2}\right)\left\{S_1+S_2\left[Q\left(\frac{v_2}{2},-z\log q\right)+\frac{\Gamma\left(\frac{r+1}{2}+\frac{v_2}{2}\right)}{\Gamma\left(\frac{r+1}{2}\right)\Gamma\left(\frac{v_2}{2}\right)}q^z\sum_{n=0}^{\infty}\frac{T_n\left(\frac{v_2}{2},q\right)}{z^{n+1}}\right]\right\}$$

$$\sim\frac{e^{-\frac{\delta^2}{2}}}{\sqrt{\pi}}\sum_{r=0}^{\infty}\frac{\delta^2}{r!}2^{\frac{r}{2}-1}\Gamma\left(\frac{r+1}{2}\right)\left\{S_1+S_2Q\left(\frac{v_2}{2},-z\log q\right)+S_2\frac{\Gamma\left(\frac{r+1}{2}+\frac{v_2}{2}\right)}{\Gamma\left(\frac{r+1}{2}\right)\Gamma\left(\frac{v_2}{2}\right)}q^z\sum_{n=0}^{\infty}\frac{T_n\left(\frac{v_2}{2},q\right)}{z^{n+1}}\right\}$$

$$\sim\frac{e^{-\frac{\delta^2}{2}}}{\sqrt{\pi}}\sum_{r=0}^{\infty}\frac{\delta^2}{r!}2^{\frac{r}{2}-1}\Gamma\left(\frac{r+1}{2}\right)S_1+\frac{e^{-\frac{\delta^2}{2}}}{\sqrt{\pi}}\sum_{r=0}^{\infty}\frac{\delta^2}{r!}2^{\frac{r}{2}-1}\Gamma\left(\frac{r+1}{2}\right)S_2Q\left(\frac{v_2}{2},-z\log q\right)$$

$$+\frac{e^{-\frac{\delta^2}{2}}}{\sqrt{\pi}}\sum_{r=0}^{\infty}\frac{\delta^2}{r!}2^{\frac{r}{2}-1}\Gamma\left(\frac{r+1}{2}\right)S_2\frac{\Gamma\left(\frac{r+1}{2}+\frac{v_2}{2}\right)}{\Gamma\left(\frac{r+1}{2}\right)\Gamma\left(\frac{v_2}{2}\right)}q^z\sum_{n=0}^{\infty}\frac{T_n\left(\frac{v_2}{2},q\right)}{z^{n+1}}$$

Which is an asymptotic expansion for the non-central *t*-distribution .

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