On Characterization of the Logistic Distribution

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Abstract

In this paper , we prove some theorems that characterize the logistic distribution with possible application of the characterization theorem is included.

Key words: Logistic distribution, Characterizations, Exponential distribution, Laplace distribution, Pareto distribution, Homogeneous Differenatial Equation.

1 Introduction

The importance of the logistic distribution is already been included in many areas of human endeavor

Balakrishnan and Leung(1988) drived the pdf of Type I generalized logistic distribution, as montioned in [3]. Olpade [4] discussed some properties of this distribution.

In this paper, we prove some theorems that will charactrize the logistic, that relate it to other probability distributions.

2 Logistic Distribution[1]

The logistic distribution is a continuous probability distribution. Its distribution function is the logistic function, which appearsin logistic regression and feedforward neural net works. It resembles the normal distribution in shape but how heavier tails. Its probability density function is given as:

$$f(x;\mu,s)=\frac{e^{-(x-\mu)/s}}{s(1+e^{-(x-\mu)/s})^2}, \quad -\infty<x<\infty \quad ,\mu\in\Re, \quad s>0 \qquad \textbf{(2.1)}$$

And the distribution function is

$$F(x;\mu,s)=\frac{1}{1+e^{-(x-\mu)/s}}, \quad -\infty<x<\infty \qquad \textbf{(2.2)}$$

If $\mu=0$, $s=1$, we get

$$F(x;0,1)=\frac{1}{1+e^{-x}}, \quad -\infty<x<\infty \qquad (2.3)$$

Which is a special case of **Type *I*** generalized logistic distribution [4],[5].

$$F(x;b)=\frac{1}{(1+e^{-x})^b}, \quad -\infty<x<\infty, \quad b>0 \qquad \textbf{(2.4)}$$

Also, this type is called the skew-logistic distribution which has the following pdf

$$f(x;b)=\frac{e^{-x}}{(1+e^{-x})^2}, \quad -\infty<x<\infty \qquad \textbf{(2.5)}$$

There are **Type *II***, **Type *III*, Type *IV*** generalized logistic distributions which are listed by Johson et al[5] as follows:

Type *II*

$$F(x;b)=1-\frac{e^{-bx}}{(1+e^{-x})^b}, \quad -\infty<x<\infty, \quad b>0 \qquad \textbf{(2.6)}$$

Type *III*

$$f(x;b)=\frac{1}{B(b,b)}\frac{e^{-bx}}{(1+e^{-x})^{2b}}, \quad -\infty<x<\infty, \quad b>0 \qquad \textbf{(2.7)}$$

Now, if we consider $b=1$, then we get a pdf of logistic distribution **(2.5)** and the equastion **(2.6)** will be turned into **(2.3)**.

Type *IV*

$$f(x;\alpha,\beta)=\frac{1}{B(\alpha,\beta)}\frac{e^{-\alpha x}}{(1+e^{-x})^{\alpha+\beta}}, \quad -\infty<x<\infty, \quad \alpha,\beta>0 \qquad \textbf{(2.8)}$$

If $\alpha=\beta=b$, then we get the pdf of **Type *III*.** And if $\beta=b=1$, then we get the pdf of logistic distribution (2.5).

3 Some Characterizations

We prove the following theorems

Theorem 1: Let X be acontinuous distributed random variable with probability density function $f_X(x)$. Then the random variable $Y=-Ln\frac{e^{-x}}{1-e^{-x}}$ is a logistic random variable if and only if X follows an exponential distribution with $\lambda=1$.

Proof: The probability density function of an exponential random variable X with λ is as follows [2]:

$$f_X(x)=\lambda e^{-\lambda x}, \quad 0<x<\infty \qquad \textbf{(3.1)}$$

And if $\lambda=1$, then

$$f_X(x)=e^{-x}, \quad -\infty<x<\infty$$

And since $Y=-Ln\frac{e^{-x}}{1-e^{-x}}$, then $0<y<\infty$ and

$$x = Ln\frac{1+e^{-y}}{e^{-y}}, \quad |J| = \frac{1}{1+e^{-y}} \text{ so}$$

$$f_Y(y) = |J| f_X(y) = \frac{e^{-y}}{(1+e^{-y})^2}, \quad -\infty < y < \infty$$

Conversely, suppose that the random variable Y is a logistic random variable, then the distribution function of X is

$$F_X(x) = P(X \le x) = P(Ln\frac{1+e^{-y}}{e^{-y}} \le x) = P(y \le Ln(e^x - 1)) = 1 - e^{-x}$$

Which is the distribution function of an exponential distribution with $\lambda = 1$.

Theorem 2: Let X be a continuous distributed random variable with probability density function $f_X(x)$. Then the random variable $Y = -Ln\frac{\frac{1}{2}e^{-x}}{1-\frac{1}{2}e^{-x}}$ is a logistic random variable if and only if X follows a laplace distribution with $\alpha = 0, \beta = 1$.

Proof: The probability density function of a laplace random variable with α and β **is as** follows **[2]:**

$$f_X(x;\alpha,\beta) = \frac{1}{2\beta}\exp(\frac{-|x-\alpha|}{\beta}), \quad -\infty < x < \infty \qquad \textbf{(3.1)}$$

And if $\alpha = 0$ and $\beta = 1$, then this function has the following form

$$f_X(x) = \frac{1}{2}e^{-x}, \quad 0 < x < \infty$$

Now, since

$$Y = -Ln\frac{\frac{1}{2}e^{-x}}{1-\frac{1}{2}e^{-x}}$$

Then

$$x = -Ln\frac{1+e^{-y}}{2e^{-y}}, \quad |J| = \frac{1}{1+e^{-y}}$$

And

$$f_Y(y) = |J| f_X(y) = \frac{e^{-y}}{(1+e^{-y})^2}, \quad 0 < y < \infty$$

We note that laplace random variable is relatef with logistic random variable over the range $(0,\infty)$.
Conversely, suppose that the random variable Y is a logistic random variable, then the distribution function of X is

$$F_X(x) = P(X \le x) = P(Ln\frac{1+e^{-y}}{2e^{-y}} \le x)$$
$$= P(y \le Ln(2e^x - 1)) = 1 - \frac{1}{2}e^{-x}, \quad x \ge 0$$

Which is the distribution function of a laplace random variable.

Theorem 3: Let X be a continuous distributed random variable with probability density function $f_X(x)$**.** Then the random variable $Y = -Ln(X^p - 1)$ is a logistic random variable if and only if X follows a pareto distribution with $b = 1$ **and** p.

Proof: The probability density function of a laplace random variable with b and p is as follows**[2]:**

$$f_X(x;b,p) = \frac{pb^p}{X^{p+1}}, \quad x > b \qquad \textbf{(3.3)}$$

And if $b = 1$ and p, then this function has the following form

$$f_X(x;1,p) = \frac{p}{X^{p+1}}, \quad x > 1$$

Now, since

$$Y = -Ln(X-1)$$

Then

$$x = (1+e^{-y})^{1/p}, \quad |J| = \frac{e^{-y}}{p(1+e^{-y})^{1-\frac{1}{p}}}$$

So

$$f_Y(y) = |J| f_X(y) = \frac{e^{-y}}{(1+e^{-y})^2}, \quad -\infty < y < \infty$$

We note that a logistic random variable over the range $(-\infty,\infty)$ is related with pareto random variable over the range $(1,\infty)$ in this way $y = \begin{cases} \infty & if\ x = 1 \\ -\infty & if\ x = \infty \end{cases}$

Conversely, suppose that the random variable Y is a logistic random variable, then the distribution function of X is

$$F_X(x) = P(X \le x)$$

$$= P((1+e^{-y})^{1/p} \le x)$$
$$= (1+e^{-y})^{-1}\Big]_{-Ln(x^p-1)}^{\infty}$$
$$= 1 - \frac{1}{X^p}, \quad x > 1$$

Which is the distribution function of pareto random variable over the range $(1,\infty)$.

Theorem 4: The random variable X is logistic with probability density function given in the equation**(2.5)** if and only if satisfies the homogeneous differenatial equation

$$(1+e^{-x})f' + (e^{-x}-1)f = 0 \qquad (3.4)$$

Proof: If X is a logistic random variable which has the probability density function in **(2.5)** and its differentiation

$$f'(x) = \frac{2e^{-2x}}{(1+e^{-x})^3} - \frac{e^{-x}}{(1+e^{-x})^2} \tag{3.5}$$

Then it is clear to show that the equation **(3.4)** is satisfied. Conversely, if f in **(2.5)** is satisties the equation**(3.4)**, the we get its solution as follows

$$Y = f_X(x) = \frac{1}{(1+e^{-x})} + c \tag{3.6}$$

The value of c is as follows $c = \frac{e^{-x}-1}{(1+e^{-x})^2}$, that means c is not an Arbitrary constant, which makes Y a density function.

Possiple Application of Theorem 4: From equation **(3.4),** We get

$$\begin{aligned} x &= Ln\frac{f'-f}{f-f'} \\ &= Ln\frac{F''-F'}{F'-F''} \end{aligned} \tag{3.7}$$

In [4], we note that there is another form which is different fromthis equation for the same distribution**.**

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