On pair wise \( \alpha \)-continuous and pair wise pre continuous mappings

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abstract

[A.S.mashhour , I.A.hasanein and S.N.el-deeb] in 1983 studied several of \( \alpha \)-continuous and \( \alpha \)-open mapping in topological spaces in this search we show that results similar th these in bitopolological spaces.

Introduction

Let \( X,Y,Z \) be topological spaces on which no separation axioms are assumed unless explicitly stated , let \( S \) be a subset of \( X \) , the closure (resp. interior ) of \( S \) will be denoted by \( \text{cl}(S) \) (resp. \( \text{int}(S) \)) . a subset of \( S \) of \( X \) is called \( \alpha \)-set [5] (resp. semi-open set[3] , pre open set [4]) if \( S \subseteq \text{int}(\text{cl}(\text{int}(S))) \) (resp. \( S \subseteq \text{cl}(\text{int}(S)) \) , \( S \subseteq \text{int}(\text{cl}(S)) \) , the complement of an \( \alpha \)-set (resp. semi-open set, preopen set) is called \( \alpha \)-closed (resp. semi-closed , pre closed ) the space of all \( \alpha \)-set(semi-open ,pre open ) is denoted \( \alpha(X)(\text{resp. SO(X),PO(X)}) \) .it is clear that each \( \alpha \)-set is semi-open and pre open and the converse is not true.

A mapping \( f:X \to Y \) is called almost continuous [7] if for each \( x \in X \) and each open neighborhood \( V \) of \( f(x) \) there exist an open neighborhood \( U \) of \( x \) such that \( f(U) \subseteq \text{int}(\text{cl}(V)) \) , and it is called \( \theta \)-continuous if \( f(U) \subseteq \text{cl}(V) \) , a mapping \( f:X \to Y \) is called seif the inverse image of each open set in \( Y \) is mi-open set in \( X \) and \( f \) is called \( \alpha \)-continuous if the inverse image of each open
set is an $\alpha$-open set in $X$ [2], and if it is called $\alpha$-open if the image of each open set is $\alpha$-open set $Y$.

2- Pair wise $\alpha$-continuity

Definition 2-1: a bitopological space $X$ is an empty set $X$ with two topologies $t_1, t_2$ defined on it, in other word is the triple $(X, t_1, t_2)$ such that $(X, t_1)$ and $(X, t_2)$ are two topologies on the same set $X$.

Definition 2-2: let $(X, t_1, t_2)$ and $(Y, p_1, p_2)$ are two bitopological spaces a mapping $f: X \rightarrow Y$ is said to be pair wise $\alpha$-continuous iff the induced maps $g: (X, t_1) \rightarrow (Y, p_1)$ and $h: (X, t_2) \rightarrow (Y, p_2)$ are $\alpha$-continuous mapping.

Definition 2-3: a mapping $f$: $(X, t_1, t_2) \rightarrow (Y, p_1, p_2)$ is said to be pair wise $\theta$-continuous iff the induced maps $g: (X, t_1) \rightarrow (Y, p_1)$ and $h: (X, t_2) \rightarrow (Y, p_2)$ are $\theta$-continuous.

Definition 2-4: a mapping $f$: $(X, t_1, t_2) \rightarrow (Y, p_1, p_2)$ is said to be pairwise semi-continuous iff the induced maps $g: (X, t_1) \rightarrow (Y, p_1)$ and $h: (X, t_2) \rightarrow (Y, p_2)$ are semi-continuous.

Definition 2-5: a mapping $f$: $(X, t_1, t_2) \rightarrow (Y, p_1, p_2)$ is said to be pair wise pre-continuous iff the induced maps $g: (X, t_1) \rightarrow (Y, p_1)$ and $h: (X, t_2) \rightarrow (Y, p_2)$ are pre-continuous.

Definition 2-6: a mapping $f$: $(X, t_1, t_2) \rightarrow (Y, p_1, p_2)$ is said to be pair wise almost-continuous iff the induced maps $g: (X, t_1) \rightarrow (Y, p_1)$ and $h: (X, t_2) \rightarrow (Y, p_2)$ are almost-continuous.
Theorem 2-1: let \( f: (X, t_1, t_2) \rightarrow (Y, p_1, p_2) \) be a mapping, then the following statements are equivalent:

i- \( f \) is pair wise \( \alpha \)-continuous

ii- each \( x \in X \) and each open set \( V \subset Y \) containing \( f(x) \) there exist \( W \subset X \) such that \( x \in W, f(W) \subset V \).

iii- the inverse image of each closed set in \( Y \) is \( \alpha \)-closed set.

Proof: (i \( \rightarrow \) ii)

since \( f \) is pair wise \( \alpha \)-continuous, then the induced maps \( g:(X, t_1) \rightarrow (Y, p_1) \)
and \( h:(X, t_2) \rightarrow (Y, p_2) \) are \( \alpha \)-continuous mapping and by definition of \( \alpha \)-continuity in topological space the result exist.

Proof (ii \( \rightarrow \) iii)

if we let (ii) then we have that \( f:X \rightarrow Y \) is an \( \alpha \)-continuous and by lating the complement of the open set and the complement of \( \alpha \)-open set we have (iii).

Corollary 2-1:

Let \( f: (X, t_1, t_2) \rightarrow (Y, p_1, p_2) \) be pair wise \( \alpha \)-continuous then

i- \( f(\text{cl}(A)) \subset \text{cl}(f(A)) \) for each \( A \in \text{PO}(X) \)

ii- \( \text{cl}(f^{-1}(M)) \subset f^{-1}(\text{cl}(M)) \) for each \( M \in \text{PO}(X) \).

Proof: see [1] and [6]

Theorem 2-2: every pair wise \( \alpha \)-continuous mapping is pair wise \( \theta \)-continuous.

Proof: let \( f: (X, t_1, t_2) \rightarrow (Y, p_1, p_2) \) be pair wise \( \alpha \)-continuous this implies that the induced maps \( g:(X, t_1) \rightarrow (Y, p_1) \) and \( h:(X, t_2) \rightarrow (Y, p_2) \) are \( \alpha \)-continuous
Now to prove that \( f \) is pair wise \( \theta \)-continuous we must prove that the
induced maps \( g \) and \( f \) are \( \theta \)-continuous

Now we have \( g:(X,t_1) \rightarrow (Y,p_1) \) is \( \alpha \)-continuous. let \( x \in X \) and \( V \subset Y \) be an
open set containing \( g(x) \), by theorem (1-1) in [ ] \( \text{cl}(\text{int}(\text{cl}(g^{-1}(V)))) \subset g^{-1}(\text{cl}(V)) \),
since \( g \) is \( \alpha \)-continuous we have \( g^{-1}(V) \subset \text{int}(\text{cl}(g^{-1}(V))) \subset \text{cl}(g^{-1}(\text{int}(\text{cl}(V)))) \subset g^{-1}(\text{cl}(V)) \), put \( \text{int}(\text{cl}(g^{-1}(V))) = U \), so \( U \)
is neighborhood of \( x \) such that \( \text{cl}(U) \subset g^{-1}(\text{cl}(V)) \), namely \( g(\text{cl}(U)) \subset V \) there
for \( g \) is \( \theta \)-continuous, similarly we can prove that \( h \) is \( \theta \)-continuous Which
is mean that \( f \) is pair wise \( \theta \)-continuous.

Remark(2-1): it is clear that the class of pair wise \( \alpha \)-continuity contains the
class of pair wise continuity but it is contained in the class of pair wise \( \theta \)-
continuity. pair wise pre-continuity and the concepts of pair wise \( \alpha \)-
continuity and pair wise almost-continuity independent. the following
diagram summarized the above discussion:

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pair wise semi-continuous
  ↑
pair wise continuity    ⇔ ⇔ ⇔ pair wise \( \alpha \)-continuity ⇔ pair wise pre-
continuity
  ↓
pair wise almost-continuity    ⇒    pair wise \( \theta \)-continuity
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4
The example given below show that the converse of these implication are not true in general.

Example 2-1: a mapping \( f: (X,t_1,t_2) \rightarrow (Y,p_1,p_2) \) such that if \( (X,t_1) \) and \( (X,t_2) \) are indiscrete spaces and \( (Y,p_1),(Y,p_2) \) are discrete spaces then \( g:(X,t_1) \rightarrow (Y,p_1) \) is pre continuous but it is not \( \alpha \)-continuous , similarly \( h:(X,t_2) \rightarrow (Y,p_2) \) , which is mean that \( f \) is pair wise pre continuous but it is not pair wise \( \alpha \)-continuous .

Example 2-2: let \( Y=X=\{1,2,3\} , t_1=\{X,\emptyset, \{1\},\{3\},\{1,3\}\} \),
\( t_2=\{X,\emptyset, \{2\},\{3\}\} \) such that \( g(1)=1,g(2)=2=g(3) \) and \( h(1)=h(2)=2,h(3)=3 \) and \( p_1,p_2 \) are discrete spaces then \( g:(X,t_1) \rightarrow (Y,p_1) \) and \( h:(X,t_2) \rightarrow (Y,p_2) \) are semi-continuous but not \( \alpha \)-continuous which is mean that \( f: (X,t_1,t_2) \rightarrow (Y,p_1,p_2) \) is pair wise semi-continuous but not pair wise \( \alpha \)-continuous.

Example 2-3: let \( Y=X=\{a,b,c\} , t_1=\{X,\emptyset, \{a\}\} \), \( t_2=\{X,\emptyset, \{b\}\} \) such that \( g:(X,t_1) \rightarrow (Y,p_1) \) is defined by \( g(a)=g(b)=a,g(c)=c \) and \( h:(X,t_2) \rightarrow (Y,p_2) \) is defined by \( h(a)=h(b)=b,h(c)=c \) , then \( g,h \) are \( \alpha \)-continuous but not continuous which is mean that \( f: (X,t_1,t_2) \rightarrow (Y,p_1,p_2) \) is pair wise \( \alpha \)-continuous but not pair wise continuous.

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