An Asymptotic Expansion for the Non-Central F-Distribution

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Abstract

A new asymptotic expansion is derived for the non-central F-distribution $F(F \backslash m_1, m_2, \lambda)$, which is suitable for large $\frac{m_1}{2}$, small $\frac{m_2}{2}$, $r \in N$ and $0<q<1$, where $Q\left(\frac{m_2}{2}, -k\log q\right) = \frac{\Gamma\left(\frac{m_2}{2}, -k\log q\right)}{\Gamma\left(\frac{m_2}{2}\right)}$ is the incomplete Gamma function ratio and $k = \left(\frac{m_1}{2} + r\right) + \frac{\left(\frac{m_2}{2} - 1\right)}{2}$. This form has some advantages over previous asymptotic expansions in this region of the parameter space in which H_n depends on all three parameters $\frac{m_1}{2}$, $\frac{m_2}{2}$ and q. The advantage of this new expansion is that an algorithm based on it can be more easily tuned for particular accuracy requirements and for particular parameter ranges.

الصيغة التقاربية لتوزيع F اللامركزي

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الخلاصة

صيغة تقاربية جديدة اشتقت لتوزيع F اللامركزي $F(F \backslash m_1, m_2, \lambda)$ والتي تكون مناسبة لقيمة كبيرة $\frac{m_1}{2}$ وقيمة صغيرة $\frac{m_2}{2}$ حيث $r \in N$ و $0<q<1$، عندما $Q\left(\frac{m_2}{2}, -k\log q\right) = \frac{\Gamma\left(\frac{m_2}{2}, -k\log q\right)}{\Gamma\left(\frac{m_2}{2}\right)}$ هي نسبة دالة كاما الغير كاملة حيث $k = \left(\frac{m_1}{2} + r\right) + \frac{\left(\frac{m_2}{2} - 1\right)}{2}$. هذه الصيغة تمتلك بعض الفوائد عن الصيغ التقاربية السابقة في هذا المجال لفضاء المعلمة والتي فيها H_n تعتمد على ثلاث معلمات والتي هي $\frac{m_1}{2}$ ، $\frac{m_2}{2}$ وكذلك q. ان فائدة هذه الصيغة هو ان اية خوارزمية مبنية عليها تكون اكبر دقة في تعيين مدى المعلمات.

1. Introduction

The non-central F-distribution $F(F' \backslash m_1, m_2, \lambda)$ is defined by (Henry, 1959; Walkk, 2001) as the form .

If X_1 and X_2 are independent random variables and X_1 is a non central chi-square distribution with m_1 degrees of freedom and non centrality parameter λ and X_2 is a central chi-square distribution with m_2 degrees of freedom then the variable $F' = \frac{X_1/m_1}{X_2/m_2}$ is said to have a non-central F-Distribution with m_1, m_2 degrees of freedom (positive integers) and non-central parameter $\lambda \geq 0$ and we write $F' \sim F_{m_1, m_2, \lambda}$.

The distribution function is given by

$$F(F' \backslash m_1, m_2, \lambda) = \sum_{r=0}^{\infty} \frac{e^{-\lambda/2}\left(\frac{\lambda}{2}\right)^r}{r!} I_q\left(\frac{m_1}{2}+r, \frac{m_2}{2}\right), \tag{1}$$

where $q = \frac{\frac{m_1 F'}{m_2}}{1+\frac{m_1 F'}{m_2}}$, $I_q\left(\frac{m_1}{2}+r, \frac{m_2}{2}\right) = \frac{B_q\left(\frac{m_1}{2}+r, \frac{m_2}{2}\right)}{B\left(\frac{m_1}{2}+r, \frac{m_2}{2}\right)}$ is incomplete Beta function.

$$\therefore F(F' \backslash m_1, m_2, \lambda) = \sum_{r=0}^{\infty} \frac{e^{-\lambda/2}\left(\frac{\lambda}{2}\right)^r}{r!} \frac{\Gamma\left(\frac{m_1}{2}+r+\frac{m_2}{2}\right)}{\Gamma\left(\frac{m_1}{2}+r\right)\Gamma\left(\frac{m_2}{2}\right)} \int_0^q t^{\frac{m_1}{2}+r-1}(1-t)^{\frac{m_2}{2}-1} dt$$

which is the cumulative distribution function (c.d.f.) of non-central F-Distribution

The asymptotic expansion was studied by other researchers who worked in our field which as the following.

The asymptotic expansion for the ratio of two Gamma functions derived by (Fields, 1966; Luke, 1969; Frenzen, 1987).

A special case of the asymptotic expansion for a ratio of products of gamma functions derived by (Biihring, 2000). He generalized a formula which was stated by (Dingle, 1973), first proved by (Paris, 1992) and recently reconsidered by (Oliver, 1995).

The special functions and their approximations had been studied by (Luke, 1969). The incomplete laplace integrals: Uniform asymptotic expansion with application to the incomplete beta function studied by (Temme, 1987). Asymptotic expansions of the Coefficients in asymptotic series solutions of linear differential equations, Methods and applications of analysis derived by (Olver, 1994).

The Uniform asymptotic expansions of integrals studied by (Temme, 1995) by using examples of stieltjies work on asymptotic of special functions. A Uniform asymptotic expansion for the Jacobi polynomials with explicit remainder derived by (Wong & Zhang, 1996). The valid asymptotic expansions for the maximum likelihood estimator of the parameter of a stationary, Gaussian, strongly dependent process studied by (Lieberman, Roussean & Zucker, 2003). A uniform asymptotic expansions for incomplete Rieman zeta functions derived by (Dunster, 2004). The uniform asymptotic expansions for hypogeometric functions with large parameters studied by (Daalhuis, 2005).

2. Derivation of an Asymptotic Expansion for the Non Central *F*-Distribution.

We derive an asymptotic expansion of $F(F' \setminus m_1, m_2, \lambda)$, through two stages:

First Stage:

We shall derive the asymptotic expansion of $\dfrac{\Gamma\left(\frac{m_1}{2}+r\right)}{\Gamma\left(\frac{m_1}{2}+r+\frac{m_2}{2}\right)}$ where $\frac{m_1}{2}$, $\frac{m_2}{2}>0$, $\frac{m_1}{2}\geq\frac{m_2}{2}$ and $r \in N$, we start from the Beta function $B\left(\frac{m_1}{2}+r,\frac{m_2}{2}\right)$, which has the formula.

$$B\left(\frac{m_1}{2}+r,\frac{m_2}{2}\right)=\frac{\Gamma\left(\frac{m_1}{2}+r\right)\Gamma\left(\frac{m_2}{2}\right)}{\Gamma\left(\frac{m_1}{2}+r+\frac{m_2}{2}\right)}=\int_0^1 t^{\frac{m_1}{2}+r-1}(1-t)^{\frac{m_2}{2}-1}dt. \tag{2}$$

Then by using the substitution $t=e^{-u}$ and $dt=-e^{-u}\,du$ we obtain

$$B\left(\frac{m_1}{2}+r,\frac{m_2}{2}\right)=\int_0^\infty e^{-\left(\frac{m_1}{2}+r\right)u}\left(1-e^{-u}\right)^{\frac{m_2}{2}-1}du. \tag{3}$$

And using the fact that $\left(1-e^{-u}\right)=e^{-u/2}\,2\mathrm{Sinh}\,\frac{u}{2}$ we have

$$B\left(\frac{m_1}{2}+r,\frac{m_2}{2}\right)=\int_0^\infty e^{-ku}u^{\frac{m_2}{2}-1}\left[\mathrm{Sinh}\left(\frac{u}{2}\right)\Big/\left(\frac{u}{2}\right)\right]^{\frac{m_2}{2}-1}du, \tag{4}$$

where $k=\left(\frac{m_1}{2}+r\right)+\dfrac{\left(\frac{m_2}{2}-1\right)}{2}$. Now expand $\left[\mathrm{Sinh}\left(\frac{u}{2}\right)\Big/\frac{u}{2}\right]^{\frac{m_2}{2}-1}$ in powers of u^2 as $u\to\infty$

$$\left[Sinh\left(\frac{u}{2}\right)\Big/\left(\frac{u}{2}\right)\right]^{\frac{m_2}{2}-1}=\left[\sum_{n=0}^{\infty}\frac{u^{2n}}{(2n+1)!2^{2n}}\right]^{\frac{m_2}{2}-1}=\left(\sum_{n=0}^{\infty}h_n u^{2n}\right)^{\frac{m_2}{2}-1}\sim\sum_{n=0}^{\infty}C_n u^{2n}, \tag{5}$$

Let $h_n = \left[\sum_{n=0}^{\infty} \frac{1}{(2n+1)!2^{2n}} \right]^{\frac{m_2}{2}-1}$. The last quality follows by (Didonate & Morris,1992).

Where C_n are the expansion coefficients of $\left[\mathrm{Sinh}\left(\frac{u}{2}\right) \Big/ \frac{u}{2} \right]^{\frac{m_2}{2}-1}$, and which can be expressed of the generalized Bernoulli polynomials (Luke,1969), $C_n = B_{2n}^{1-\frac{m_2}{2}} \frac{\left(\frac{1-\frac{m_2}{2}}{2} \right)}{(2n)!}$.By substitution equation (5) in (4) we get

$$B\left(\frac{m_1}{2}+r,\frac{m_2}{2}\right) \sim \int_0^{\infty} e^{-ku} u^{\frac{m_2}{2}-1} \sum_{n=0}^{\infty} C_n u^{2n} du$$

$$B\left(\frac{m_1}{2}+r,\frac{m_2}{2}\right) \sim \sum_{n=0}^{\infty} C_n \int_0^{\infty} e^{-ku} u^{\frac{m_2}{2}+2n-1} du .$$

And using Watson's Lemma we obtain the asymptotic expansion

$$\frac{\Gamma\left(\frac{m_1}{2}+r\right)}{\Gamma\left(\frac{m_1}{2}+r+\frac{m_2}{2}\right)} \sim \frac{1}{k^{\frac{m2}{2}}} \sum_{n=0}^{\infty} C_n \frac{\Gamma\left(\frac{m_2}{2}+2n\right)}{\Gamma\left(\frac{m_2}{2}\right)} \left(\frac{1}{k}\right)^{2n} . \tag{6}$$

Second Stage:

In this stage we derive the asymptotic expansion of $I_q\left(\frac{m_1}{2}+r,\frac{m_2}{2}\right)$,

$$I_q\left(\frac{m_1}{2}+r,\frac{m_2}{2}\right) = \frac{\Gamma\left(\frac{m_1}{2}+r+\frac{m_2}{2}\right)}{\Gamma\left(\frac{m_1}{2}+r\right)\Gamma\left(\frac{m_2}{2}\right)} \int_0^q t^{\frac{m_1}{2}+r-1} (1-t)^{\frac{m_2}{2}-1} dt , \tag{7}$$

when $\frac{m_1}{2}, \frac{m_2}{2} > 0$, $\frac{m_1}{2} \geq \frac{m_2}{2}$, $0 < q < 1$ and $r \in N$ and then transform the expression for it as the same in equation (2) and changing integration terms, to obtain.

$$I_q\left(\frac{m_1}{2}+r,\frac{m_2}{2}\right) = \frac{\Gamma\left(\frac{m_1}{2}+r+\frac{m_2}{2}\right)}{\Gamma\left(\frac{m_1}{2}+r\right)\Gamma\left(\frac{m_2}{2}\right)} \int_{-\log q}^{\infty} e^{-ku} u^{\frac{m_2}{2}-1} \left[\mathrm{Sinh}\left(\frac{u}{2}\right) \Big/ \left(\frac{u}{2}\right) \right]^{\frac{m_2}{2}-1} du , \tag{8}$$

where as before $k = \left(\frac{m_1}{2}+r\right) + \frac{\left(\frac{m_2}{2}-1\right)}{2}$, by using (5) we have

$$I_q\left(\frac{m_1}{2}+r,\frac{m_2}{2}\right) \sim \frac{\Gamma\left(\frac{m_1}{2}+r+\frac{m_2}{2}\right)}{\Gamma\left(\frac{m_1}{2}+r\right)\Gamma\left(\frac{m_2}{2}\right)} \sum_{n=0}^{\infty} C_n \int_{-\log q}^{\infty} e^{-ku} u^{2n+\frac{m_2}{2}-1} du . \tag{9}$$

Let w=ku, then $u=\frac{1}{k}w$ and $du=\frac{1}{k}dw$, so from (9) we have

$$I_q\left(\frac{m_1}{2}+r,\frac{m_2}{2}\right)\sim\frac{\Gamma\left(\frac{m_1}{2}+r+\frac{m_2}{2}\right)}{\Gamma\left(\frac{m_1}{2}+r\right)\Gamma\left(\frac{m_2}{2}\right)}\sum_{n=0}^{\infty}C_n\int_{-k\log q}^{\infty}e^{-w}\left(\frac{1}{k}w\right)^{2n+\frac{m_2}{2}-1}\frac{1}{k}dw$$

$$=\frac{\Gamma\left(\frac{m_1}{2}+r+\frac{m_2}{2}\right)}{\Gamma\left(\frac{m_1}{2}+r\right)\Gamma\left(\frac{m_2}{2}\right)}\sum_{n=0}^{\infty}C_n\frac{\Gamma\left(\frac{m_2}{2}+2n\right)}{k^{2n+\frac{m_2}{2}}}\frac{1}{\Gamma\left(\frac{m_2}{2}+2n\right)}\int_{-k\log q}^{\infty}e^{-w}w^{2n+\frac{m_2}{2}-1}dw$$

$$\sim\frac{\Gamma\left(\frac{m_1}{2}+r+\frac{m_2}{2}\right)}{\Gamma\left(\frac{m_1}{2}+r\right)\Gamma\left(\frac{m_2}{2}\right)k^{\frac{m_2}{2}}}\sum_{n=0}^{\infty}\frac{C_n}{k^{2n}}\Gamma\left(\frac{m_2}{2}+2n\right)Q\left(\frac{m_2}{2}+2n,-k\log q\right),\qquad(10)$$

where Q(.,.) is incomplete gamma function ratio.
We can proceed by using the recurrence relations for $Q(.,.)$ to express $Q\left(\frac{m_2}{2}+2n,-k\log q\right)$ in terms of $Q\left(\frac{m_2}{2},-k\log q\right)$. This gives

$$I_q\left(\frac{m_1}{2}+r,\frac{m_2}{2}\right)\sim Q\left(\frac{m_2}{2},-k\log q\right)+R\left(\frac{m_1}{2}+r,\frac{m_2}{2},q\right).\qquad(11)$$

where we have use the formula (6) to cancel out the factors multiplying Q, the other term $R\left(\frac{m_1}{2}+r,\frac{m_2}{2},q\right)$ is a double summation over n and $2n$ residual terms obtained by expressing $Q\left(\frac{m_2}{2}+2n,-k\log q\right)$ in terms of $Q\left(\frac{m_2}{2},-k\log q\right)$.

To obtain the asymptotic expansion we require to reordering this sum. First we write (8) in the form

$$I_q\left(\frac{m_1}{2}+r,\frac{m_2}{2}\right)=\frac{\Gamma\left(\frac{m_1}{2}+r+\frac{m_2}{2}\right)}{\Gamma\left(\frac{m_1}{2}+r\right)\Gamma\left(\frac{m_2}{2}\right)}$$

$$\left[\int_{-\log q}^{\infty}e^{-ku}\left(\left[2\mathrm{Sinh}(u/2)\right]^{\frac{m_2}{2}-1}u^{\frac{m_2}{2}-1}\right)du+\int_{-\log q}^{\infty}e^{-ku}u^{\frac{m_2}{2}-1}du\right].\qquad(12)$$

Integrate the first integral by parts twice as follows :

$$\int_{-\log q}^{\infty}e^{-ku}\left(\left[2\mathrm{Sinh}(u/2)\right]^{\frac{m_2}{2}-1}-u^{\frac{m_2}{2}-1}\right)du$$

$$=\frac{1}{k^2}\int_{-\log q}^{\infty}e^{-ku}\frac{d^2}{du^2}\left(\left[2Sinh(u/2)\right]^{\frac{m_2}{2}-1}-u^{\frac{m_2}{2}-1}\right)du$$

$$+\frac{q^k}{k}\left[\left[2\mathrm{Sinh}(u/2)\right]^{\frac{m_2}{2}-1}-u^{\frac{m_2}{2}-1}+\frac{1}{k}\frac{d}{du}\left(\left[\left[2\mathrm{Sinh}(u/2)\right]^{\frac{m_2}{2}-1}-u^{\frac{m_2}{2}-1}\right]\right)\right]\Bigg|_{u=-\log q}. \qquad (13)$$

In the integral in (13) we now subtract the second term $\left(C_1 u^{\frac{m_2}{2}+1}\right)$ in expansion of $\left[2\mathrm{Sinh}(u/2)\right]^{\frac{m_2}{2}-1}$ and add a corresponding integral so that the integral in (13) becomes

$$\int_{-\log q}^{\infty} e^{-ku}\frac{d^2}{du^2}\left[\left[2\mathrm{Sinh}(u/2)\right]^{\frac{m_2}{2}-1}-u^{\frac{m_2}{2}-1}-C_1 u^{\frac{m_2}{2}+1}\right]du+\frac{\Gamma\left(\frac{m_2}{2}+2\right)}{\Gamma\left(\frac{m_2}{2}\right)}C_1\int_{-\log q}^{\infty} e^{-ku}u^{\frac{m_2}{2}-1}du \,.(14)$$

The first of these integrals is then integrated by parts twice producing two further integrated terms evaluated at $u=-\log q$ and an integral of a fourth derivative. In this integral, a further term from the expansion of $\left[2\mathrm{Sinh}(u/2)\right]^{\frac{m_2}{2}-1}$, $C_2 u^{\frac{m_2}{2}+3}$ is subtracted from the differentiated part and a corresponding integral added on separately. This procedure is continued indefinitely. The separate integrals starting from the ones on the right of (12) and (14) add together to give $Q\left(\frac{m_2}{2},-k\log q\right)$ as in (11) so that

$$I_q\left(\frac{m_1}{2}+r,\frac{m_2}{2}\right)\sim Q\left(\frac{m_2}{2},-k\log q\right)+\frac{\Gamma\left(\frac{m_1}{2}+r+\frac{m_2}{2}\right)}{\Gamma\left(\frac{m_1}{2}+r\right)\Gamma\left(\frac{m_2}{2}\right)}q^k\sum_{n=0}^{\infty}\frac{H_n\left(\frac{m_2}{2},q\right)}{k^{n+1}}, \qquad (15)$$

where $H_n\left(\frac{m_2}{2},q\right)=\frac{d^n}{du^n}\left(\left[2\mathrm{Sinh}(u/2)\right]^{\frac{m_2}{2}-1}-\sum_{\ell=0}^{n/2}C_\ell u^{2\ell+\frac{m_2}{2}-1}\right)\Bigg|_{u=-\log q}$

$$=\frac{d^n}{du^n}\left[\sum_{\ell=\frac{n}{2}+1}^{\infty}C_\ell u^{2\ell+\frac{m_2}{2}-1}\right]\Bigg|_{u=-\log q},$$

where $n/2$ in the summation is to be interpreted as largest integer $\le n/2$ as in integer division. The quantities H_n satisfy the simple recurrence formula $H_{2n+1}=\frac{d}{du}H_{2n}$,

$$H_{2n}=\frac{d}{du}H_{2n-1}-C_n u^{\frac{m_2}{2}-1}\frac{\Gamma\left(2n+\frac{m_2}{2}\right)}{\Gamma\left(\frac{m_2}{2}\right)}. \qquad (16)$$

We can express $H_n\left(\frac{m_2}{2},q\right)$ directly in terms of $\frac{m_2}{2}$ and q, for example, $H_0\left(\frac{m_2}{2},q\right)=\left(1/\sqrt{q}-\sqrt{q}\right)^{\frac{m_2}{2}-1}-(-\log q)^{\frac{m_2}{2}-1}$. However, for q close to 1, evaluation H_n

in this way can lead to large rounding errors on subtraction, and so $H_n\left(\frac{m_2}{2}, q\right)$ is better evaluated from its power series expansion in u. Now, when we substitute the formula (15) in equation (1), we get

$$F(F' \backslash m_1, m_2, \lambda) \sim \sum_{r=0}^{\infty} \frac{e^{-\lambda/2}(\lambda/2)^r}{r!} \left[Q\left(\frac{m_2}{2}, -k \log q\right) + \frac{\Gamma\left(\frac{m_1}{2} + r + \frac{m_2}{2}\right)}{\Gamma\left(\frac{m_1}{2} + r\right)\Gamma\left(\frac{m_2}{2}\right)} q^k \sum_{n=0}^{\infty} \frac{H_n\left(\frac{m_2}{2}, q\right)}{k^{n+1}} \right]$$

$$\sim \sum_{r=0}^{\infty} \frac{e^{-\lambda/2}(\lambda/2)^r}{r!} Q\left(\frac{m_2}{2}, -k \log q\right) + \sum_{r=0}^{\infty} \frac{\Gamma\left(\frac{m_1}{2} + r + \frac{m_2}{2}\right)}{\Gamma\left(\frac{m_1}{2} + r\right)\Gamma\left(\frac{m_2}{2}\right)} \frac{e^{-\lambda/2}(\lambda/2)^r}{r!} q^k \sum_{n=0}^{\infty} \frac{H_n\left(\frac{m_2}{2}, q\right)}{k^{n+1}}.$$

By using the identity $\sum_{r=0}^{\infty} \frac{e^{-\lambda}\lambda^r}{r!} = 1$, we get

$$F(F' \backslash m_1, m_2, \lambda) \sim Q\left(\frac{m_2}{2}, -k \log q\right) + \frac{1}{\Gamma\left(\frac{m_2}{2}\right)} \sum_{r=0}^{\infty} \frac{\Gamma\left(\frac{m_1}{2} + r + \frac{m_2}{2}\right) e^{\frac{-\lambda}{2}} \left(\frac{\lambda}{2}\right)^r q^k}{\Gamma\left(\frac{m_1}{2} + r\right) r!} \sum_{n=0}^{\infty} \frac{H_n\left(\frac{m_2}{2}, q\right)}{k^{n+1}}. \quad (17)$$

which is an asymptotic expansion for the non-central F-distribution.

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