An Asymptotic Expansion for the Non-Central Chi-square Distribution

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Abstract

We derive an asymptotic expansion for the non-central chi-square distribution as $\frac{U}{2} \to \infty$,where $U = \sum_{i=1}^{m} X_i^2$ is the non-central chi-square variable with m degree of freedom $(U \geq 2)$, $\lambda \geq 0$ and $r \in N$.

Through certify from advantage of this asymptotic, we found out that an algorithm based on this expansion can be more easily tuned for particular accuracy requirements and for particular parameter ranges.

الصيغة التقاربية لتوزيع مربع كاي اللامركزي

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الخلاصة

نشتق الصيغة التقاربية لتوزيع مربع كاي اللامركزي عندما $\frac{U}{2}$ تقترب إلى المالانهاية حيث أن $U = \sum_{i=1}^{m} X_i^2$ يمثل متغير مربع كاي اللامركزي بدرجة حرية m $(U \geq 2)$ ، $\lambda \geq 0$ وكذلك $r \in N$.

من خلال التأكد من جدوى هذا التقريب وجد بأن أي خوارزمية مبنية على هذه الصيغة تكون اكبر دقة في تعيين مدى المعلمات .

1. Introduction

The non-central chi-square distribution is defined by(Henry,1959;Mood,Graybill &Boes,1974) as the following

If $X_1, X_2, ..., X_m$ are independently distributed and X_i is $N(\mu_i, 1)$ then the random variable $U = \sum_{i=1}^{m} X_i^2$ is called a non-central chi-square variable with m degree of freedom. We call $\delta = \left(\sum_{i=1}^{m} \mu_i^2 \right)^{1/2}$ the non centrality parameter of the distribution and $\lambda = \delta^2$ is the non-central parameter .The distribution function is given by

$$F(U \setminus m, \lambda) = \sum_{r=0}^{\infty} \frac{e^{\frac{-\lambda}{2}} \left(\frac{\lambda}{2} \right)^r}{r!} \; P\left(\frac{m}{2} + r, \frac{U}{2} \right) \qquad U \geq 0, \lambda \geq 0 \quad .$$

$$= \sum_{r=0}^{\infty} \frac{e^{\frac{-\lambda}{2}} \left(\frac{\lambda}{2} \right)^r}{r!} \; \frac{\gamma\left(\frac{m}{2} + r, \frac{U}{2} \right)}{\Gamma\left(\frac{m}{2} + r \right)} \quad ,$$

where $P\left(\frac{m}{2} + r, \frac{U}{2} \right) = \dfrac{\int_0^{\frac{U}{2}} e^{-t} t^{\frac{m}{2}+r-1} \, dt}{\int_0^{\infty} e^{-t} t^{\frac{m}{2}+r-1} \, dt}$ is incomplete gamma function ratio, $\gamma\left(\frac{m}{2} + r, \frac{U}{2} \right) = \int_0^{\frac{U}{2}} e^{-t} t^{\frac{m}{2}+r-1} \, dt$ is incomplete gamma function and $\Gamma\left(\frac{m}{2} + r \right) = \int_0^{\infty} e^{-t} \; t^{\frac{m}{2}+r-1} dt$ is gamma function .

$$F(U \setminus m, \lambda) = \sum_{r=0}^{\infty} \frac{e^{\frac{-\lambda}{2}} \left(\frac{\lambda}{2} \right)^r}{r! \; \Gamma\left(\frac{m}{2} + r \right)} \int_0^{\frac{U}{2}} t^{\frac{m}{2}+r-1} \; e^{-t} \; dt \tag{1}$$

which is the cumulative distribution function of non-central chi-square distribution .

A non-central chi-square distribution studied by other researchers such as(Sankar-an,1963;Ruben,1974;Venables,1975;Siegel,1979;Anderson,1981;Alam,1982;Ennis&John son,1993) whose worked in our field by taking the different subject of this distribution such as :Approximations to the non central chi-square distribution, Estimation of the noncentrality parameter of a chi-square distribution, Maximum likelihood estimation in the non-central chi-distribution ,calculation of confidence intervals for non-central distributions,… .We introduce many researches who used the asymptotic expansion in our field as the form

The uniform asymptotic expressions for saddle point integrals–Application to a probability distribution occurring in noise theory derived by (Rice,1968) and the uniform asymptotic expansion for a class of polynomials biorthogonal on the unit circle derived by (Temme,1986) . An incomplete laplace integrals: uniform asymptotic expansion with application to the incomplete Beta function studied by(Temme,1987) and the uniform asymptotic expansions of Laguerre polynomials derived by (Frenzen & Wong,1988) .The generalized incomplete gamma functions studied by(Chaudhry & Zubair,1994)as well as the asymptotic and closed from of generalized incomplete gamma function derived by (Chaudhry,Temme and Veling,1996) .A uniform asymptotic expansions for Meixner polynomials derived by(Jin & Wong,1998) and the asymptotic of the hypergeometric function introduced by (Jones,2001).

2- Motivating example the incomplete gamma function

Consider the following incomplete gamma function :

$$\gamma\left(\frac{m}{2}+r,\frac{U}{2}\right)=\int_0^{\frac{U}{2}} e^{-t}t^{\frac{m}{2}+r-1}\,dt \quad ,$$

This function is defined like the gamma function, except that the second variable $\frac{U}{2}$ appears in the upper limit of integration.Obviously, $\gamma\left(\frac{m}{2}+r,0\right)=0$ and $\gamma\left(\frac{m}{2}+r,\infty\right)=\Gamma\left(\frac{m}{2}+r\right)$.Another related function is also called the incomplete Gamma function:

$$\Gamma\left(\frac{m}{2}+r,\frac{U}{2}\right)=\int_{\frac{U}{2}}^{\infty} e^{-t}\,t^{\frac{m}{2}+r-1}\,dt=\Gamma\left(\frac{m}{2}+r\right)-\gamma\left(\frac{m}{2}+r,\frac{U}{2}\right),$$

where $\Gamma\left(\frac{m}{2}+r\right)=\int_0^{\infty} e^{-t}\,t^{\frac{m}{2}+r-1}dt$ is the ordinary gamma function .

It is easy to check that $\Gamma\left(\frac{m}{2}+r,0\right)=\Gamma\left(\frac{m}{2}+r\right)$, $\Gamma\left(\frac{m}{2}+r,\infty\right)=0$.

3- Derivation of an Asymptotic Expansion for the Non-Central Chi-square Distribution

Consider the integral term in (1) called incomplete gamma function $\gamma\left(\frac{m}{2}+r,\frac{U}{2}\right)$. To find an asymptotic expansion of the $\gamma\left(\frac{m}{2}+r,\frac{U}{2}\right)$ as $\frac{U}{2}\to\infty$ by expanding the function in powers of t and integrating term by term, we have

$$\gamma\left(\frac{m}{2}+r,\frac{U}{2}\right)=\int_0^{\frac{U}{2}} t^{\frac{m}{2}+r-1}\,e^{-t}\,dt=\int_0^{\frac{U}{2}} t^{\frac{m}{2}+r-1}\left(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\ldots\right)dt$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^{\frac{U}{2}} t^{\frac{m}{2}+r+n-1}\,dt \sim \sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{U}{2}\right)^{\frac{m}{2}+r+n}}{n!\left(\frac{m}{2}+r+n\right)} .$$

Using the ratio test, we see that the series converges for all $\frac{U}{2}$ .However, the series is only useful for small $\frac{U}{2}$ (for large values of $\frac{U}{2}$,it converges very slowly).To deal with $\gamma\left(\frac{m}{2}+r,\frac{U}{2}\right)$ for $\frac{U}{2}$ large ,we proceed indirectly with $\Gamma\left(\frac{m}{2}+r,\frac{U}{2}\right)$ and $\frac{m}{2}+r$ fixed .Rather than taylor expanding the integrand in $\frac{U}{2}$ (as we did above),write

$$\gamma\left(\frac{m}{2}+r,\frac{U}{2}\right)=\Gamma\left(\frac{m}{2}+r\right)-\Gamma\left(\frac{m}{2}+r,\frac{U}{2}\right) \tag{2}$$

$$\gamma\left(\frac{m}{2}+r,\frac{U}{2}\right)=\int_0^\infty e^{-t}\ t^{\frac{m}{2}+r-1}\ dt\ -\int_{\frac{U}{2}}^\infty\ e^{-t}\ t^{\frac{m}{2}+r-1}\ dt\qquad .$$

We now take the incomplete gamma function

$$\Gamma\left(\frac{m}{2}+r,\frac{U}{2}\right)=\int_{\frac{U}{2}}^\infty\ e^{-t}\ t^{\frac{m}{2}+r-1}\ dt\qquad .$$

We shall develop the integral by repeating the integration by parts:

$$\Gamma\left(\frac{m}{2}+r,\frac{U}{2}\right)=\left[-e^{-t}\ t^{\frac{m}{2}+r-1}\right]_{\frac{U}{2}}^{\infty}\ +\left(\frac{m}{2}+r-1\right)\int_{\frac{U}{2}}^\infty\ e^{-t}\ t^{\frac{m}{2}+r-2}\ dt$$

$$=e^{\frac{-U}{2}}\left(\frac{U}{2}\right)^{\frac{m}{2}+r-1}+\left(\frac{m}{2}+r-1\right)e^{\frac{-U}{2}}\left(\frac{U}{2}\right)^{\frac{m}{2}+r-2}$$

$$+\left(\frac{m}{2}+r-1\right)\left(\frac{m}{2}+r-2\right)\int_{\frac{U}{2}}^\infty\ e^{-t}\ t^{\frac{m}{2}+r-3}\ dt$$

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$$\Gamma\left(\frac{m}{2}+r,\frac{U}{2}\right)=e^{\frac{-U}{2}}\left[\left(\frac{U}{2}\right)^{\frac{m}{2}+r-1}+\left(\frac{m}{2}+r-1\right)\left(\frac{U}{2}\right)^{\frac{m}{2}+r-2}+\ldots+\left(\frac{m}{2}+r-1\right)\ldots\right.$$

$$\left.\left(\frac{m}{2}+r-N+1\right)\left(\frac{U}{2}\right)^{\frac{m}{2}+r-N}\right]+\left(\frac{m}{2}+r-1\right)\left(\frac{m}{2}+r-2\right)\ldots\left(\frac{m}{2}+r-N\right)\int_{\frac{U}{2}}^\infty e^{-t}t^{\frac{m}{2}+r-N-1}dt\,,$$

where the remainder term

$$R_N\left(\frac{m}{2}+r,\frac{U}{2}\right)=\left(\frac{m}{2}+r-1\right)\left(\frac{m}{2}+r-2\right)\ldots\left(\frac{m}{2}+r-N\right)\int_{\frac{U}{2}}^\infty\ e^{-t}\ t^{\frac{m}{2}+r-N-1}dt$$

$$\Gamma\left(\frac{m}{2}+r,\frac{U}{2}\right)=e^{\frac{-U}{2}}\left(\frac{U}{2}\right)^{\frac{m}{2}+r}\left[\frac{1}{\left(\frac{U}{2}\right)}+\frac{\left(\frac{m}{2}+r-1\right)}{\left(\frac{U}{2}\right)^{2}}+\ldots+\frac{\left(\frac{m}{2}+r-1\right)\ldots\left(\frac{m}{2}+r-N+1\right)}{\left(\frac{U}{2}\right)^{N}}\right]^{\frac{U}{2}}$$

$$+R_N\left(\frac{m}{2}+r,\frac{U}{2}\right).$$

The integrands of successive integrals are becoming smaller in region of integration $1\le\frac{U}{2}<t$,$U\ge 2$ .However, this series generated by integration by parts does not convergent for fixed finite $\frac{U}{2}$ (by using the ratio test).But for fixed N ,the error committed by omitting $R_N\left(\frac{m}{2}+r,\frac{U}{2}\right)$ is small for large $\frac{U}{2}$,that is, as $\frac{U}{2}\to\infty$, $R_N\left(\frac{m}{2}+r,\frac{U}{2}\right)\to 0$ for fix-ed N .In order to prove this ,we estimate R_N as follows :

Assume $N>\frac{m}{2}+r-1$ for definiteness, the largest of $t^{\frac{m}{2}+r-N-1}$ in the range of integration is $\left((\frac{U}{2}\right)^{\frac{m}{2}+r-N-1}$

$$\left|\int_{\frac{U}{2}}^{\infty} e^{-t}\, t^{\frac{m}{2}+r-N-1}dt\right|\le\int_{\frac{U}{2}}^{\infty}\left|e^{-t}\, t^{\frac{m}{2}+r-N-1}\right|dt$$

$$\le\left(\frac{U}{2}\right)^{\frac{m}{2}+r-N-1}\int_{\frac{U}{2}}^{\infty} e^{-t}\,dt$$

$$=\left(\frac{U}{2}\right)^{\frac{m}{2}+r-N-1}e^{\frac{-U}{2}}=o\left(\left(\frac{U}{2}\right)^{\frac{m}{2}+r-N}e^{\frac{-U}{2}}\right)\qquad\text{as }\frac{U}{2}\to\infty\ .$$

Thus ,for sufficiently large $\frac{U}{2}$,the remainder term R_N will be small and only a few terms in the series are needed to give a reasonable approximation to $\Gamma\left(\frac{m}{2}+r,\frac{U}{2}\right)$.Series sums like that are called asymptotic expansions, and are written as:

$$\Gamma\left(\frac{m}{2}+r,\frac{U}{2}\right)\sim e^{\frac{-U}{2}}\left(\frac{U}{2}\right)^{\frac{m}{2}+r}\left(\frac{1}{\left(\frac{U}{2}\right)}+\frac{\left(\frac{m}{2}+r-1\right)}{\left(\frac{U}{2}\right)^{2}}+\frac{\left(\frac{m}{2}+r-1\right)\left(\frac{m}{2}+r-2\right)}{\left(\frac{U}{2}\right)^{3}}+\ldots\right)$$

By substitution in equation (2) ,we get

$$\gamma\left(\frac{m}{2}+r,\frac{U}{2}\right)\sim\Gamma\left(\frac{m}{2}+r\right)-e^{\frac{-U}{2}}\left(\frac{U}{2}\right)^{\frac{m}{2}+r}\left(\frac{1}{\left(\frac{U}{2}\right)}+\frac{\left(\frac{m}{2}+r-1\right)}{\left(\frac{U}{2}\right)^{2}}+\frac{\left(\frac{m}{2}+r-1\right)\left(\frac{m}{2}+r-2\right)}{\left(\frac{U}{2}\right)^{3}}+..\right)$$

as $\frac{U}{2}\rightarrow\infty$, (3)

although the series is divergent .

Now ,when we substitute the formula (3) in equation (1) ,we get

$$F(U\setminus m,\lambda)\sim\sum_{r=0}^{\infty}\frac{e^{\frac{-\lambda}{2}}\left(\frac{\lambda}{2}\right)^{r}}{r!}-\sum_{r=0}^{\infty}\frac{e^{-\left(\frac{\lambda+U}{2}\right)}\left(\frac{\lambda}{2}\right)^{r}\left(\frac{U}{2}\right)^{\frac{m}{2}+r}}{r!\;\Gamma\left(\frac{m}{2}+r\right)}\left(\frac{1}{\left(\frac{U}{2}\right)}+\frac{\left(\frac{m}{2}+r-1\right)}{\left(\frac{U}{2}\right)^{2}}+\frac{\left(\frac{m}{2}+r-1\right)\left(\frac{m}{2}+r-2\right)}{\left(\frac{U}{2}\right)^{3}}+\ldots\right).$$

Then by using the identity $\sum_{r=0}^{\infty}\frac{e^{-\lambda}\ \lambda^{r}}{r!}=1$,we have

$$F(U\setminus m,\lambda)\sim 1-\sum_{r=0}^{\infty}\frac{e^{-\left(\frac{\lambda+U}{2}\right)}\left(\frac{\lambda}{2}\right)^{r}\left(\frac{U}{2}\right)^{\frac{m}{2}+r}}{r!\;\Gamma\left(\frac{m}{2}+r\right)}\left(\frac{1}{\left(\frac{U}{2}\right)}+\frac{\left(\frac{m}{2}+r-1\right)}{\left(\frac{U}{2}\right)^{2}}+\frac{\left(\frac{m}{2}+r-1\right)\left(\frac{m}{2}+r-2\right)}{\left(\frac{U}{2}\right)^{3}}+\ldots\right)$$

as $\frac{U}{2}\rightarrow\infty$, (4)

this result gives an asymptotic expansion for the non-central chi-square distribution .

References

Alam K & Saxena L (1982) Estimation of the noncentrality parameter of a chi-square distribution .Annals of Statistics,10:1012-1016 .

Anderson D A (1981) Maximum likelihood estimation in the non-central chi distribution with unknown scale parameter .Sankhya ,Series B,43b:58-67 .

Chaudhry M A,Temme N M & Veling E J M (1996) Asymptotic and closed from of a generalized incomplete gamma function .Journal of Computational and Applied Mathematics 67 :371-379 .

Chaudhry M A & Zubair S M (1994) Generalized incomplete gamma functions with applications .Journal of Computational and Applied Mathematics 55 :99-124 .

Ennis D M & Johnson N L (1993) Non-central and central chi-square ,F and beta distribution functions as special cases of the distribution of an indefinite quadratic form, Communications in Statistics-Theory and Methods ,22:897-905 .

Frenzen C L & Wong R (1988) Uniform asymptotic expansions of Laguerre polynomials . SIAM J. Math. Anal.,19 :1232-1248 .

Henry S (1959) The analysis of variance .John Wiley and sons ,Inc. New York .

Jin X S & Wong R (1998) Uniform asymptotic expansions for Meixner polynomials. Constructive Approximation 14 :113-150 .

Jones D S (2001) Asymptotic of the hypergeometric function .Math. Methods Appl. Sci. 24 :369-389 .

Mood A M ,Graybill F A & Boes D C (1974) Introduction to the theory of statistics .3d ed. , Mc Graw-Hill , Inc. New York .

Rice S O (1968) Uniform asymptotic expansion for saddle point integrals –Application to a probability distribution occurring in noise theory. Bell System Technical Journal, 47 :1971-2013 .

Ruben H (1974) Non-central chi-square and gamma revisited.Communication in Statistics ,3 : 607-633 .

Sankaran M (1963) Approximations to the non-central chi-square distribution .Biometrika,50 :199-204 .

Siegel A F (1979) The non-central chi-squared distribution with zero degrees of freedom and testing for uniformity .Biometrika,66 :381-386 .

Temme N M (1987) Incomplete laplace integrals :uniform asymptotic expansion with application to the incomplete beta function .SIAM J. Math. Anal.,18 :1638-1663 . MR & 9f: 41036 .

Temme N M (1986) Uniform asymptotic expansion for a class of polynomials biorthogonal on the unit circle .Constr. Approx. ,2 :369-376 .

Venables W N (1975) Calculation of confidence intervals for non-central distributions . Journal of the Royal Statistical Society, Series B ,27 :406-412 .