Almost Copositive Polynomial Approximation in $L_p[-1,1], p<1$

Dr. Eman Samir Bhaya, Department of Mathematics, College of Education,

Babylon University, Babil, Hilla, Iraq.

E.mail: emanbhaya@yahoo.com

Abstract. ***For a function*** $f \in L_p[-1,1], 0<p<1$ ***with finitely many sign changes, Hu, Kopotun and Yu [5] construct a sequence of polynomials*** $p_n \in P_n$ ***which are copositive with*** f ***and such that*** $\|f-p_p\|_p \le c(p)\omega_\varphi(f,n^{-1})_p$***, where*** $\omega_\varphi(f,n^{-1})_p$ ***denotes the Ditzian-Totik modulus of continuity in*** L_p ***metric. Also it was shown that this estimate is exact in the sense that if*** f ***has at least one sign change then*** ω_φ ***can not be replaced by*** ω^2 ***if*** $0<p<1$***. In this paper we first show that almost copositive approximation improves the rate to*** $\omega_\varphi^2(f,n^{-1})_p$ ***from*** $\omega_\varphi(f,n^{-1})_p$ ***the rate for the ordinary copositive approximation. Our second theorem shows that it is impossible to obtain the first estimate interims of*** ω_φ^4***.***

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1. Introduction and definitions

Let $L_p[a,b]$ be the set of all measurable functions on $[a,b]$ such that

$\|f\|_{L_p[a,b]} < \infty$, where $\|f\|_{L_p[a,b]} := \left(\int_a^b |f(x)|^p dx \right)^{1/p}$, $0 < p < 1$.(see [1] p. 19-24)

Let P_n denote the set of all polynomials of degree $\leq n$, by N the set of natural numbers. Thought this paper the notation $C = C(a,b)$ denote constants that are depend only on a,b and is independent of every thing else, and are not necessarily the same even if they occur in the same line.

For $Y_s ;= \{y_1, y_2, \ldots, y_s, y_0 = -1 < y_1 < \ldots < y_s < 1 = y_{s+1}\}$. We denote by $\Delta_0(Y_s)$ the set of all functions $f \in L_p[-1,1]$ such that $(-1)^{s-k} f(x) \geq 0$ for $x \in [y_k, y_{k+1}], k = 0,1,2,\ldots, s$, it mean every $f \in \Delta_0(Y_s)$ has $0 \leq s < \infty$ sign changes at the points in Y_s and is nonnegative near I. A function g is said to be *copositive* with f if $f(x)g(x) \geq 0$ for all $x \in [-1,1]$.

We are interested in coapproximation function from $\Delta^0(Y_s)$ by polynomials p_n of degree$\leq n$ that are copositive with f. For $f \in L_p[-1,1]$ let

$$E_n(f)_p := \inf_{p_n \in P_n} \| f - p_n \|_p ,$$

denote the degree of *unconstrained approximation* and let

$$E_n^0(f, Y_s)_p := \inf_{p_n \in P_n \cap \Delta^0(Y_s)} \| f - p_n \|_p ,$$

be the *degree of copositive approximation* to f by algebraic polynomials of degree $\leq n$, where $\|f\|_p = \|f\|_{L_p[a,b]}$. The *degree of intertwining polynomial approximation* of functions $f \in L_p[-1,1]$ with respect to Y_s is given by

$$\widetilde{E}_n(f, Y_s)_p := \inf\{\|P - Q\|_p : P, Q \in P_n, P - f \in \Delta^0(Y_s) \text{ and } f - Q \in \Delta^0(Y_s)\},$$

We call $\{P, Q\}$ an intertwining pair of polynomials for f with respect to Y_s if $P - f, f - p \in \Delta^0(Y_s)$.For more details see [7].

We denote $J_j(n, \in) = [y_j - \Delta_n(y_j) n^{\in}, y_j + \Delta_n(y_j) n^{\in}] \cap [-1,1]$, $j = 0,1,\ldots, s+1$ and denote $O_n(Y_s, \in) = \cup_{j=1}^{s} J_j(n, \in)$ and $O_n^*(Y_s, \in) = \cup_{j=0}^{s+1} J_j(n, \in)$. If $\in = 0$ we shall also use the simpler notation $J_j = J_j(n,0)$, $O_n(Y_s) = O_n(Y_s, 0)$ and $O_n^*(Y_s) = O_n^*(Y_s, 0)$. Functions f and g are called *copositive* on $J \subset I := [-1,1]$ if $f(x) g(x) \geq 0 \forall x \in J$. Function f and g are called *almost copositive* on I with respect to Y_s if they are copositive on $I - O_n^*(Y_s)$. We say that f and g are *strongly (weakly) almost copositive* on I with respect to Y_s if they are copositive on

$I - O_n(Y_s, \in)$ where $\varepsilon < 0(\varepsilon > 0)$. In particular, if $\varepsilon = -\infty$, then strongly almost copositive functions are just copositive. Define a function class

$$(\varepsilon - alm\Delta)_n^0(Y_s) := \{f : (-1)^{s-k} f(x) \ge 0 \text{ for } x \in I - O_n^*(Y_s, \in).$$

If $s = 0$ it becomes:

$$(\varepsilon - alm\Delta)_n^0 := (\varepsilon - alm\Delta)_n^0(Y_0)$$

$$:= \{f : f(x) \ge 0 \text{ for } x \in [-1 + n^{-2+\varepsilon}, 1 - n^{-2+\varepsilon}]\},$$

the set of all strongly (weakly) almost nonnegative functions on I if $\varepsilon < 0(\varepsilon > 0)$. Again if $\varepsilon = 0$ we omit the letter ε in the notation and use $(alm\Delta)_n^0(Y_s)$ and $(alm\Delta)_n^0$ the latter is the set of almost nonnegative functions on I . If $\varepsilon = -\infty$, strongly almost nonnegative functions are just nonnegative.

We define a function class:

$(alm\Delta)_n^0(Y_s) = \{f : (-1)^{s-k} f(x) \ge 0 \text{ for } x \in I - O_n^*(Y_s)\}$. The *degree of almost copositive polynomial approximation* of $f \in L_p[-1,1] \cap \Delta^0(Y_s)$ is

$$E_n^{(0)}(f, almY_s)_p := \inf\{\|f - p\|_p : p \in P_n \cap (alm\Delta)_n^o(Y_s)\}$$

Similarly, we define $E_n^{(0)}(f, \varepsilon - almY_s)_p$ the degree of strongly (weakly) almost copositive polynomial approximation of $f \in L_p[-1,1] \cap \Delta^0(Y_s)$ by means of $p \in P_n \cap (\varepsilon - alm\Delta)_n^0(Y_s)$.

It was shown by Hu, Kopotun and Yu [5] that if f changes its sign in $(-1,1)$, ω_φ^1 being the best order of approximation:

Theorem A. *If* $f \in L_p[-1,1] \cap \Delta^0(Y_s) 0 < p < 1$ *then for every* $n \in N - \{0\}$

$$E_n^0(f, Y_s)_p \leq c(p)\omega_\varphi(f, n^{-1})_p .$$

Also it was shown that:

One can not replace $\omega_\varphi^1(f, n^{-1})_p$ by $\omega_\varphi^2(f, n^{-1})_p$ for $0 < p < 1$, where

$$\omega_\varphi^m(f, t)_p := \sup_{0<h\leq t} \left\| \Delta_{h\varphi(.)}^m(f, \cdot, [-1,1]) \right\|_p$$

is the m^{th} *Ditzian Totik modulus* of smoothness with $\varphi(x) = \sqrt{1 - x^2}$, and

$$\Delta_h^m(f, x, [-1,1]) := \left\{ \sum_{i=0}^{m} \binom{m}{i} (-1)^{m-i} f\left(x - \frac{m}{2}h + ih\right) \text{ if } x \pm \frac{m}{2}h \in [-1,1] \right\}.$$

Little is known about copositive and almost copositive approximation of functions in $L_p[-1,1] \cap \Delta^0(Y_s)$ for $1 \leq p < \infty$ and $s \geq 1$ and it seems that nothing is known in the case for $0 < p < 1$. It turns out that things become more complicated in L_p .

Now the order ω_φ^2 is impossible, we seek for a best rate. Our theorem below shows that almost copositive approximation in $L_p, 0 < p < 1$ improves the rate to $\omega_\varphi^2(f, n^{-1})_p$ from $\omega_\varphi^1(f, n^{-1})_p$:

TheoremI. *Suppose* $f \in L_p[-1,1] \cap \Delta^0(Y_s) 0 < p < 1$ *for any* $n > c(Y_s)$ *we have*

$$E_n^{(0)}(f, almY_s)_p \le c(p,s)\omega_\varphi^2(f, n^{-1})_p \qquad (1)$$

The following theorem and corollary show that (1) is exact for *0<p<1,* that is

TheoremII. *Let* Y_s *be fixed. For any given* $A > 0, 0 < p < 1$ *and sufficiently large* $n \in N$, *there exists a function* $f \in C[-1,1] \cap \Delta^0(Y_s)$ *such that for every polynomial* $p_n \in P_n$ *which is copositive with* f *on* $\left[y_s + \frac{1-y_s}{3}, 1 - \frac{1-y_s}{3}\right]$ *the following inequality holds*

$$\|f - p_n\|_p > An^\beta \omega_\varphi^4(f, n^{-1})_p \qquad (2)$$

where $\beta < \frac{p}{p+2}$.

CorolaryIII. *Let* Y_s *be fixed. For any given* $0 \le \varepsilon < 1$ *and sufficiently large* $n \in N$ *,there exists* $f \in C[-1,1] \cap \Delta^0(Y_s)$ *such that*

$$E_n^{(0)}(f, \varepsilon - almY_s)_p > ac(p,s)\omega_\varphi^4(f, n^{-1})_p.$$

2. Weak Copositive Approximation

In this section we show in theorem I that weak almost copositive approximation in $L_p, 0 < p < 1$ improves the rate to ω_φ^2 from ω_φ^1. We first need the following result from [6] :

Lemma1. *Let* $Y_s, s \geq 0$ *be given* $m \in N, \mu \geq 2m + 30, 0 < p < \infty$, *and let* $S(x)$ *be a spline of an odd order* $r = 2m + 1$ *on the knot sequence* $\left\{x_i = \cos\frac{i\pi}{n}\right\}_{i \in I_n(Y_s)}$ *where* $n > c(Y_s)$ *is such that there are at least 4 knots* x_i *in each interval* $(y_j, y_{j+1}), j = 0, \ldots, s$ *and* $I_n(Y_s) = \{1, \ldots, n\} \backslash \{i, i-1, x_i \leq y_j < x_{i-1}\}$ *for some* $1 \leq j \leq s$. *Then there exists an intertwining pair of polynomials* $\{P_1, P_2\} \subset P_{c(r)n}$ *for* S *with respect to* Y_s *such that*

$$\|P_1 - P_2\|_p^p \leq c(r, s, \min\{1, p\})^p \sum_{i=1}^{n-1} E_{r-1}\left(S, \hat{I}_i \cup \hat{I}_{i+1}\right)_p^p, \ \textit{if } o < p < \infty \qquad (3)$$

where $\hat{I}_i = [x_i, x_{i-1}]$.

Also we need the following assertion in [4]

Lemma 2. *for any* $f \in L_p(I), 0 < p < 1$ *and* $r \in N$ *we have*

$$E_n(f)_p \leq c(p)\omega_\varphi^r(f, n^{-1})_p.$$

Proof of theorem I.

Note that

$$E_n^{(0)}(f, \varepsilon - almY_s)_p^p \leq \widetilde{E}_n(f, Y_s)_p^p$$

$\leq \left\| P_1 - P_2 \right\|_p^p$, where P_1, P_2 the polynomials defined in Lemma 1. Then Lemma 1 and Lemma 2 imply

$$E_n^{(0)}\left(f, \varepsilon - almY_s\right)_p^p \leq c(p,s)^p \sum_{i=1}^{n-1} E_{r-1}\left(S, I_i^* \cup I_{i+1}^*\right)_p^p .$$

For $f \in L_p(I), 0 < p < 1$ define a quadratic spline on

$T_{k-2s} = \left\{ x_i = \cos\frac{i\pi}{k} \right\}_{i \in I_k(Y_s)}$, by $S = Tf := \sum_{i=-r+1}^{k+1} c_i N_i, c_i = d_i^{*-1} \int_{I_i^*} |f|$, d_i^* is an absolute constant $I_i^* = \left[t_i^* - \frac{d_i^*}{2}, t_i^* + \frac{d_i^*}{2} \right]$, t_i^* is an auxiliary knots in $[-1,1]$ (see [3], p. 223), to get

$$E_n^{(0)}\left(f, \varepsilon - almY_s\right)_p^p \leq c(p,s)^p \sum_{i=1}^{k-1} E_2\left(S, I_i^* \cup I_{i+1}^*\right)_p^p$$

$$\leq c(p,s) \sum_{i=1}^{k-1} \omega_\varphi^2\left(f, \left|I_i^* \cup I_{i+1}^*\right|, I_i^* \cup I_{i+1}^*\right)_p^p$$

$$\leq c(p,s) \omega_\varphi^2\left(f, n^{-1}\right)_p^p . \quad ☻$$

3. The counter example

In this section we construct the counter example described in Theorem II. We show that weakly almost copositive approximation doesn't do better than those Corollary III, in spite of larger intervals in which the restriction is relaxed.

Proof of Theorem II

Let $n \geq s+2$, $L(x)=\left(\left(x-\frac{1+y_s}{2}\right)^2-b^2\right)\prod_{j=1}^{s}(x-y_j)$ where $b<\frac{1-y_s}{6}$

is a constant, and let

$$f(x)=\begin{cases} L(x) & if\ x\notin\left[\frac{1+y_s}{6}-b,\frac{1+y_s}{6}+b\right] \\ 0 & otherewise \end{cases}$$

suppose that *(2)* is true, it means there exists a polynomial $p_n \in P_n$ such

that $p_n(x)\geq 0$ for $x\in\left[\frac{1+y_s}{6}-b,\frac{1+y_s}{6}+b\right]$ and

$\|f-p_n\|_p \leq An^{\beta}\omega_{\varphi}^4\left(f,n^{-1}\right)_p$. Let us assume $\beta \geq 0$. Note that

$\|f-L\|_p=\left(\int_{\frac{1+y_s}{2}-b}^{\frac{1+y_s}{2}+b}|L(x)|^p\,dx\right)^{1/p}$, and since $b<\frac{1+y_s}{6}<\frac{1+y_s}{2}$ we have

$\|f-L\|_p=c(p)b^{2+1/p}$, and

$$\begin{aligned}\omega_{\varphi}^4\left(f,n^{-1}\right)_p &\leq c(p)\omega_{\varphi}^4\left(f-L,n^{-1}\right)_p+c(p)\omega_{\varphi}^4\left(L,n^{-1}\right)_p \\ &\leq c(p)\|f-L\|_p+c(p)n^{-4}\left\|L^{(4)}\right\|_p \\ &\leq c(p)b^{2+1/p}+c(p)n^{-4}.\end{aligned}$$

Also by the well known inequality in [2]

$$\|p_k\|_{L_\infty[a,b]} \le c(p,k)(b-a)^{-1/p}\|p_k\|_{L_p[a,b]}, \text{ for } \; p_k \in P_k,$$

$$\|p_n - L\|_p \ge c(p)n^{-1}\left(p_n\left(\frac{1+y_s}{2}\right) - L\left(\frac{1+y_s}{2}\right)\right)$$

$$\ge -c(p)n^{-1}L\left(\frac{1+y_s}{2}\right)$$

$$= c(p)n^{-1}b^2 \prod_{j=1}^{s}\left(\frac{1+y_s}{2} - y_j\right)$$

$$\ge c(p)n^{-1}b^2.$$

Therefore

$$c(p)n^{-1}b^2 \le \|p_n - f\|_p + \|f - L\|_p$$

$$\le An^\beta \omega_\varphi^4\left(f, n^{-1}\right)_p + c(p)b^{2+1/p}$$

$$\le c(p)n^\beta b^{2+1/p} + c(p)n^{-4} + c(p)b^{2+1/p}$$

$$\le c(p)n^\beta b^{2+1/p} + c(p)n^{-4}.$$

This implies the inequality

$$c(p)n^{-1}b^2 - n^\beta b^{2+1/p} \le c(p)n^{-4}.$$

Now let $b = n^{-1-\frac{\beta}{p}}$, then the last inequality implies

$$n^{4-3-\frac{2\beta}{p}} - n^{\beta\left(-1-\frac{\beta}{p}\right)\left(2+\frac{1}{p}\right)} \le c(p)$$

and

$$n^{4-3-\frac{2\beta}{p}-\beta} \le c(p).$$

But this can not be true for sufficiently large n, since condition on β and p in the theorem imply $4 > -3 + 2\beta + 2\beta \frac{1}{p} + \beta$. ☻

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