Euler line in graph theory

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Abstract:
In this paper an application of Euler line in hilla bridge and
proof theorem in printer's problem / De Bruijn cycle are
performance and the situation were r=3 and all possible of
subsequence its means all Euler line in directed graph are taken.

1-Introduction:
The basic idea of graphs were introduced in 18th century by the
great swiss mathematician . He used graphs to
solved the famous k"onigsberg bridge problem.
German city of k"onigsberg(now it is Russian Kaliningrad)
Was situated on the river pregel.it had a park situated on the
banks of the river and two islands .mainland and islands were
joined by seven bridges.A problem was whether it was possible
to take a walk through the town in such a way as to cross over
every bridge once, and only once.
A graph is a set of points ( we call them vertices or nodes)
connected by lines ( edges or arcs).For more details see [1],[2]
and [6].

2-.Definition:[1][2][3]
1-A graph G=(V, E) consist of two sets ,set V of vertices and E
of edges such that each e C G can be identified with a pair (u,v)
of vertices in V. the vertices u and v are known as end points of
E.
2-A walk (w) in G is an alternative sequence of vertices and
edges
Repeation of vertices and edges are allowed awalk is said to
be an open walk if initial and end vertices are different
otherwise w is closed.
3-A trail in G is an open walk without repeation of edges .
4-A path in G is an open walk without repeation of vertices .
A graph G is said to be an Euler graph if there is a closed trail which covers all the edges of G.

6. The degree of a vertex is the number of edges incident on it. A graph is regular if all of its vertices have the same degree. i.e., if \( d(v) = k \), \( \forall v \in V \) then G is a k-regular.

8. We say that G is connected if there is a path between each pair of vertices in G, if G is not connected it is a disconnected graph.

The Königsberg bridge problem is perhaps the best-known example in graph theory. It was a long-standing problem until solved by Leonhard Euler (1707-1783) in 1736, by means of a graph. Euler wrote the first paper ever in graph theory and thus became the originator of the theory of graph, as well as of the rest of topology. The problem depicted in fig (3-1).

Two islands C and D formed by the Pregel river in Königsberg (then the capital of East Prussia but now renamed Kaliningrad and in West Soviet Russia), were connected to each other and to the banks A and B with seven bridges, as shown in fig (2-10).

The problem was to start at any of the four land areas of the city A, B, C, or D, walk over each of the seven bridges exactly once, and return to the starting point (without swimming across the river).

Euler represented this situation by means of a graph, as shown in fig (3-2). The vertices represent the land areas and the edges represent the bridges.

Euler proved that a solution for this problem does not exist.
(3-2)-graph of Königsberg bridge problem

4: Euler graphs:[2][4]

Graph theory was born in 1736 with Euler famous paper in which he solved the Königsberg problem. In the same paper, Euler posed (and then solved) a more general problem: in what type of graph $G$ is it possible to find a closed walk running through every edge of $G$ exactly once? Such a walk is now called an Euler walk, and such a graph is called an Euler graph. More formally:

If some closed walk in a graph contains all the edges of the graph, then the walk is called an Euler walk and the graph an Euler graph.

By its very definition, a walk is always connected since the Euler line (which is a walk) contains all the edges of the graph. An Euler graph is always connected, except for any isolated vertices the graph may have. Since isolated vertices do not contribute anything to the understanding of an Euler graph, do not have any isolated vertices and are there for connected.

5: Theorem:[2][3]

A given connected graph $G$ is an Euler graph if and only if vertices of $G$ are of even degree.

6: Hilla bridge:[5]

In this research, we can explain that the Hilla bridges are Euler graphs. Now looking at the fig(6-1) and the fig(6-2).

The two areas of the city $A$ and $B$ were connected to each other and walk over each of the five bridges exactly twice and return to the starting point without swimming across the river. Now we can see that there is a closed trail which covers all the edges of $G$ and we can easily see that the graph is connected and the
vertices of the graph Hilla bridges are of even degree, so by theorem or by definition, it is Euler graph or it is have Euler line.

(6-1)-Hilla bridge

(6-2)-graph of hilla bridges
7: Theorem: - [1][2][3]

How long a cyclic sequence of digits 0 & 1 such that no subsequence (segment) of r bits of 0 & 1 will appear more than once in a cyclic sequence?

Proof:
Tele printer's problem / De Bruijn cycle
We know that there was $2^r$ possible subseq of r-bits so it is clear that the length of such a cyclic sequence cannot be $>2^r$ we want to construct a cyclic sequence of digit 0 & 1 of length $2^r$ such that no subseq $^r$ (segment) of r bits of 0 & 1 will repeat in the cycle.

We shall construct on Euler digraph as follows:
Let us consider a graph $G=(V,E)$ such that all the (r-1) tuples are vertices of $G$. It is clear that, in $G$ there are $2^{r-1}$ vertices.
$P=α_1α_2.....α_{r-1}$ with $α_i = 0$ or 1 $∀i=1....(r-1)$
Now, draw two edges from p to two vertices say q1 & q2 & we label these edges as p0 & p1 i.e $α_1α_2........α_{r-1} \ 0$ & $α_1α_2........α_{r-1} \ 1$

Let us take $q_1=(α_1α_2........α_{r-1} \ 0)$

and $q_2=(α_1α_2........α_{r-1} \ 1)$

Finally draw such two edges from each vertex in G Thus , $d^{−}(V)=2 \ ∀ \ v \in V$, Note that if $p=(00.....0)$ or $(11.....1)$ then we get self-loop at $p$.
Similarly, for each $v \in V$, we get $d^{−}(V)=2$ thus, $d^{+}(v)$ for each $v \in V$. Also,$\sum_{i=1}^{2^{r-1}} d^{+}(V_i)=2^{r-1}$

$\sum_{i=1}^{2^{r-1}} 2=2^{r-1} \iff e=2^r$

It is clear that the digraph $G$ will be an Euler digraph.

Let $T$ denote an Euler line in $G$ & it contains All $2^r$ edges.
Now, consider a seq^n by taking first digit from each edge consecutively in T then it will be the required cyclic seq^n of length 2'.

Ex:

\[
\begin{array}{c}
\bullet & a_1 (a_2 \ldots a_{r-1}, a_r) & \bullet & (a_1 a_3 \ldots a_2 \ldots a_r) a_{r+1} & \bullet \\
p & q & x \\
\end{array}
\]

\[
a_1 \ldots a_{r-1} a_2 a_3 \ldots a_{r-1} a_r a_3 a_4 \ldots a_{r+1}
\]

Note that if p, q, & x are as in the figure then the (r-1) tuple of the trail of first edges will be the same as (r-1) tuple of head of the next edge in T. Further note that any two edges have different labeling.

So, in acyclic seq^n any subseq^n of r-bits will not repeat.

Now for r=3 bit we can see that there is 16 Euler line. Such that it is not repeat.

1)-\{15-16-4-5-6-7-8-9-2-3-1-10-11-12-13-14\}.
2)-\{15-16-4-12-8-9-2-3-1-10-11-5-6-7-13-14\}.
3)-\{15-16-4-12-13-3-1-10-11-5-6-7-8-9-2-14\}.
4)-\{15-16-4-12-8-11-5-6-7-13-3-1-10-9-2-14\}.
5)-\{15-16-4-5-6-7-8-9-10-11-12-13-3-1-2-14\}.
6)-\{15-16-4-5-6-7-8-11-12-13-3-1-9-2-14\}.
7)-\{15-16-4-5-6-7-13-3-1-10-11-12-8-9-2-14\}.
8)-\{15-16-4-12-8-9-10-11-5-6-7-13-3-1-2-14\}.
9)-\{15-16-1-10-11-5-6-7-8-9-2-3-4-12-13-14\}.
10)-\{15-16-1-2-3-4-12-8-9-10-11-5-6-7-13-14\}.
11)-\{15-16-1-10-11-5-6-7-8-9-2-3-4-12-13-14\}.
12)-\{15-16-1-10-9-2-3-4-5-6-7-8-11-12-13-14\}.
13)-\{15-16-1-2-3-4-5-6-7-8-9-10-11-12-13-14\}.
14)-\{15-16-1-10-11-12-8-9-2-3-4-5-6-7-13-14\}.
15)-\{15-16-1-10-11-12-8-9-2-3-4-5-6-7-13-14\}.
16)-\{15-16-1-10-11-12-13-3-4-5-6-7-8-9-2-14\}.
8: Conclusion:
1- Now looking at the graph of the Königsberg bridges (fig. 3-2), we find that not all its vertices are of even degree. Hence it is not an Euler graph. Thus it is not possible to walk over each of the seven bridges exactly once and return to the starting point. One often encounters Euler lines in various puzzles. The problem common to these puzzles is to find how a given picture can be drawn in one continuous line without retracing and without lifting the pencil from the paper.
2- and we can see that the graph Hilla bridges its Euler graph and its have Euler line.
3- And if we take r = 4 in Tele printor's problem /De Bruijn cycle then there is 32 Euler line and if r = 5 there is 64 Euler line so on.
Reference:
1- Narsingh Deo, "graph theory with applications to engineering and computer science.prentice", Hall of India pvt.Ltd, New Delhi, 1999.
4- Charles Mullins "Applications of Elementary Graph Theory", May 2002, mullinse@asms1x.dsc.k12.ar.us.

5- أحمد صباح الجنابي "أثر طرق النقل البري على نمو المستوطنات البشرية في محافظة بابل دراسة في جغرافية النقل", رسالة ماجستير ادب بعداد 2003.