The Non Wandering Points of Henon map $H_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a - by - x^2 \\ x \end{pmatrix}$

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Abstract

In this work we study the non wandering point of Henon map.The main goal of this work is to prove theorems on the Henon map $H_{a,b}$, $b > 0$ .Thus we divide the plane into three regions such that the union of this three regions covers the plane. We prove there is no non wandering point for Henon map $H_{a,b}$, where $b > 0$ and $a < \frac{-(1+b)^2}{4}$**.**

Introduction

About 30 years ago the French astronomer –mathematician Michel-Henon was searched for a simple two-dimensional map possessing special properties of more complicated systems .The result was a family of maps denoted by $H_{a,b}$ given by $H_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 - ax^2 + y \\ bx \end{pmatrix}$ where a , b are real numbers .These maps defined in above are called Henon maps [3]. In this work we, care this form of Henon map $H_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a - by - x^2 \\ x \end{pmatrix}$ [1] .Also we study the non wandering point for Henon map $H_{a,b}$ where $a < \frac{-(1+b)^2}{4}$ by using trapping region.

<u>Definition (1-1) [3]</u> Let V be a subset of R^2, and v_0 be any element in R^2.Consider $F: V \longrightarrow R^2$ be a map. Further more assume that the first partials of the coordinate maps f and g of F exist at v_0 .The **differential of F** at v_0 is the linear map **DF(v_0)** defined on R^2 by **DF(v_0)** = $\begin{pmatrix} \frac{\partial f}{\partial x}(v_0) & \frac{\partial f}{\partial y}(v_0) \\ \frac{\partial g}{\partial x}(v_0) & \frac{\partial g}{\partial y}(v_0) \end{pmatrix}$,

for all v in R^2 .The determinant of DF(v_0) is called the **Jacobian of F** at v_0 and is denoted by J=det DF(v_0) .

Example (1-2)[3]

Let F: $R^2 \longrightarrow R^2$ be given by F$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ a\sin(x)+by \end{pmatrix}$.To find DF$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ we have for $f=y$ and $g=a\sin(x)+by$ that $\frac{\partial f}{\partial x}$=0, $\frac{\partial f}{\partial y}$=1, $\frac{\partial g}{\partial x}$= $a\cos(x)$, $\frac{\partial g}{\partial y}$=b, so DF$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$=$\begin{pmatrix} 0 & 1 \\ a\cos(x_0) & b \end{pmatrix}$ for all $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ in R^2, J=$-a\cos(x_0)$

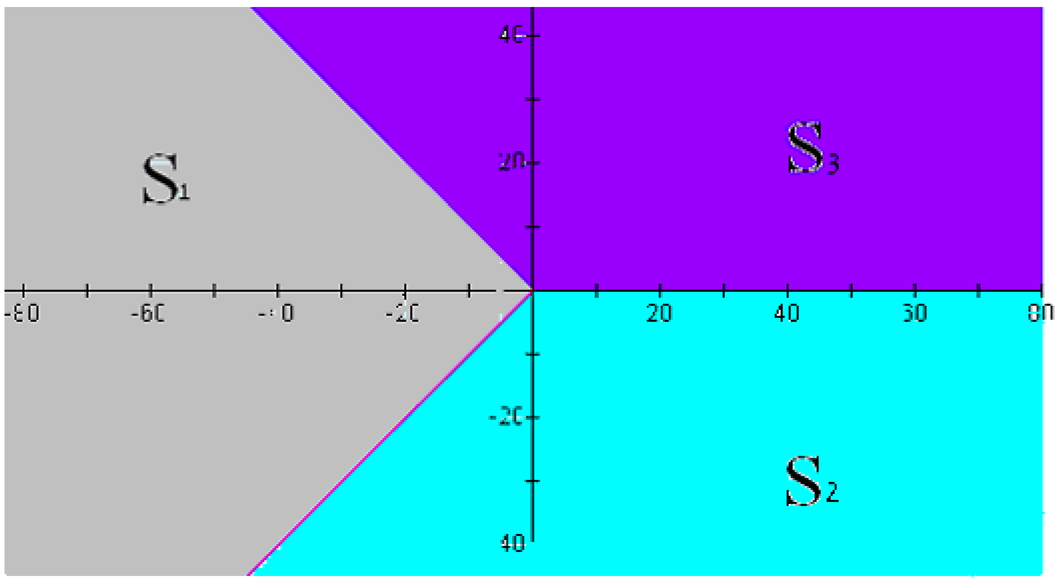
2- The Non Wandering Point of Henon Map Where $b>0$ and $a<-\frac{(1+b)^2}{4}$.

The main goal of this section is to prove two theorems on the Henon map $H_{a,b}$,$b>0$. To prove these theorems , we divide the plane into three regions such that the union of this three regions covers the plane. we suppose $a_0 = \frac{-(1+b)^2}{4}$, prove some lemmas with respect to parameters a,b and regions till we get the main purpose ,the regions are the following **:**

S_1= $\{\begin{pmatrix} x \\ y \end{pmatrix}: x,y \in R, x \le -|y| \}$

$S_2 = \{ \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in R, x \geq -|y|, y \leq 0 \}$

$S_3 = \{ \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in R, x \geq -|y|, y > 0 \}$



Fig(9):Region S_1 ,S_2 ,S_3

Definition (2-1)[1]

Let F: $R^n \longrightarrow R^n$ be a map .A closed region Q $\subset R^n$ is trapping region for F if F(Q) is contained in the interior of Q.

Lemma (2-2)

Let $a < a_0$, $b > 0$.Then S_1 is trapping region for $H_{a,b}$.

Proof: clearly S_1 is closed region .To show that $H_{a,b}(S_1) \subset S_1^0$.

Let $\begin{pmatrix} x \\ y \end{pmatrix} \in H_{a,b}(S_1)$, then there exists $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ in S_1, such that $H_{a,b}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ so

$x = a - by_0 - x_0^2, y = x_0$, since $x_0 \in R$ we have two cases :

case1: $y_0 \geq 0$ then $|y_0| = y_0, x_0 \leq -y_0$, since $b > 0$, we have $by_0 \leq -bx_0$. (2.1)

Thus $a + by_0 - x_0{}^2 \leq a - bx_0 - x_0{}^2$. (2.2)

Since $-by_0$ is a negative real number we have $-by_0 < by_0$ hence $a - by_0 - x_0{}^2 < a + by_0 - x_0{}^2$. (2.3)

Now from (2.2) and (2.3) we get $x < a - bx_0 - x_0{}^2$. (2.4)

Since $a < a_0$ from (2.4), we get $x < -\frac{(1+|b|)^2}{4} - bx_0 - x_0{}^2$

$$= -[\frac{(1+|b|)^2}{4} + (1+|b|)x_0 + x_0{}^2] + x_0$$

$$= -[\frac{1+|b|}{2} + x_0]^2 + x_0 .$$

So $x < x_0$, that is $x < y$. On the other hand $x_0 < 0$, so $|y| = -y, x < -|y|$, hence $\begin{pmatrix} x \\ y \end{pmatrix} \in S_1{}^0$,that is $H_{a,b}(S_1) \subset S_1{}^0$.

Case2: $y_0 < 0$ then $|y_0| = -y_0$,so $x_0 \leq y_0$, since $b > 0$, we have $-by_0 \leq -bx_0$ (2.5)

thus $a - by_0 - x_0{}^2 \leq a - bx_0 - x_0{}^2$. (2.6)

Now as the same as above case , we get $x < x_0$, $x_0 \leq y_0 < 0$ so $|y| = -y$ thus $x < -|y|$, hence $\begin{pmatrix} x \\ y \end{pmatrix} \in S_1{}^0$, that is $H_{a,b}(S_1) \subset S_1{}^0$. □

Lemma (2-3)

Let $a < a_0$,$b > 0$.Then S_2 is trapping region for $H_{a,b}{}^{-1}$.

Proof: clearly, S_2 is closed region. To show that $H_{a,b}{}^{-1}(S_2) \subset S_2{}^0$, let $\begin{pmatrix} x \\ y \end{pmatrix} \in H_{a,b}{}^{-1}(S_2)$,then there exists $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ in S_2, such that $H_{a,b}{}^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$, so

$x = y_0$, $y = \dfrac{a - x_0 - y_0^{\ 2}}{b}$, since $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in S_2$ we have $x_0 \geq -|y_0|$ and $y_0 \leq 0$ that is

$x_0 \geq y_0$. (2.7)

Now since $b > 0, a < a_0$ from (2.7) we have :

$$\begin{aligned} by = a - x_0 - y_0^{\ 2} &< -\frac{(1+b)^2}{4} - x_0 - y_0^{\ 2} \\ &< -\frac{(1+b)^2}{4} - y_0 - y_0^{\ 2} \\ &= -[\frac{(1+b)^2}{4} + y_0 + by_0 - by_0 + y_0^{\ 2}] \\ &= -[\frac{(1+b)^2}{4} + (1+b)y_0 + y_0^{\ 2}] + by_0 \\ &= -[\frac{(1+b)}{2} + y_0]^2 + by_0 . \end{aligned}$$

Hence $by < by_0$,since $b > 0$, we get $y < y_0$,that is $y < x$, since $x = y_0 \leq 0$, we get $|y| = -y$, $y < x$. Hence $-|y| < x$, so $\begin{pmatrix} x \\ y \end{pmatrix} \in S_2^{\ 0}$, hence $H_{a,b}^{\ -1}(S_2) \subset S_2^{\ 0}$. □

Proposition (2-4)[2]

Let $a < a_0$, $b > 0$.

(i) $H_{a,b}(S_1 \cup S_3) \subset S_1^{\ 0}$

(ii) $\langle x_n \rangle$ is strictly decreasing sequence along $H_{a,b}$ -orbits and $\left\| H_{a,b}^{\ n} \begin{pmatrix} x \\ y \end{pmatrix} \right\| \longrightarrow \infty$ as $n \longrightarrow \infty$, for all $\begin{pmatrix} x \\ y \end{pmatrix}$ in S_1.

(iii) $H_{a,b}^{\ -1}(S_2 \cup S_3) \subset S_2^{\ 0}$.

(iv) $\langle |y_{-n}| \rangle$ is strictly increasing sequence along $H_{a,b}^{-1}$-orbits and $\left\| H_{a,b}^{-n}\begin{pmatrix} x \\ y \end{pmatrix} \right\| \longrightarrow \infty$ as $n \longrightarrow \infty$ for all $\begin{pmatrix} x \\ y \end{pmatrix}$ in S_2.

Proof: See [2]

Remark: From Proposition (2-4) we note that $S_1 \cup S_3$ is trapping region for $H_{a,b}$ and $S_2 \cup S_3$ is trapping region for $H_{a,b}^{-1}$.

<u>Theorem (2-5)</u>

Let $a < a_0(b)$, $b > 0$, the Henon map $H_{a,b}$ has no periodic point for any period, that is $P_{er_n}(H_{a,b}) = \phi$.

Proof: Suppose that there exists a periodic point $\begin{pmatrix} x \\ y \end{pmatrix}$ for $H_{a,b}$ of period n, where $n \in N$ then must be in R^2 and from the partition with respect to S_1, S_2 and S_3. Since $\begin{pmatrix} x \\ y \end{pmatrix} \in R^2$ then either $\begin{pmatrix} x \\ y \end{pmatrix}$ in $S_1 \cup S_2$ or S_3. If $\begin{pmatrix} x \\ y \end{pmatrix} \in S_1 \cup S_2$, then by lemma(2-4)(ii) $\langle x_n \rangle$ is strictly decreasing sequence along $H_{a,b}$-orbits in S_1 and $\left\| H_{a,b}^{n}\begin{pmatrix} x \\ y \end{pmatrix} \right\| \longrightarrow \infty$ as $n \longrightarrow \infty$ for all $\begin{pmatrix} x \\ y \end{pmatrix}$ in S_1 and from lemma(2-4)(iv) $\langle |y_{-n}| \rangle$ is strictly increasing sequence along $H_{a,b}^{-1}$ orbits in S_2 and $\left\| H_{a,b}^{-n}\begin{pmatrix} x \\ y \end{pmatrix} \right\| \longrightarrow \infty$ as $n \longrightarrow \infty$ for all $\begin{pmatrix} x \\ y \end{pmatrix}$ in S_2 that means there is no finite orbit for any point in $S_1 \cup S_2$, so $\begin{pmatrix} x \\ y \end{pmatrix}$ has no finite orbit which is contradiction .

If $\begin{pmatrix} x \\ y \end{pmatrix} \in S_3$, by lemma (2-4)(i) $H_{a,b}(S_3) \subset S_1^{\,0}$ and $S_1^{\,0} \subset S_1$, thus S_3 maps into S_1 and by our supposition $\begin{pmatrix} x \\ y \end{pmatrix}$ is periodic, so $H_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$ is a periodic point in S_1 which is contradiction . So there is no periodic point for $H_{a,b}$ in $S_1 \cup S_2 \cup S_3$ and since $S_1 \cup S_2 \cup S_3 = R^2$. We get $H_{a,b}$ has no periodic point for any period .□

Definition (2-6)[5]

Let F: $R^2 \longrightarrow R^2$ be a map. A point $\begin{pmatrix} p \\ q \end{pmatrix}$ is called **non wandering** provided that for every neighborhood U of $\begin{pmatrix} p \\ q \end{pmatrix}$, there is an integer $n > 0$ such that $F^n(U) \cap U \neq \phi$ .Thus ,there is a point $\begin{pmatrix} r \\ s \end{pmatrix} \in U$ with $F^n \begin{pmatrix} r \\ s \end{pmatrix} \in U$. The set of all non wandering points for F is called the **non wandering set** and is denoted by $\Omega(F)$.

Definition (2-7)[4]

Let F: $R^2 \longrightarrow R^2$ be a map. Let $K^+(F)$ denotes the set of points in R^2 with bounded forward orbits .Let $K^-(F)$ denotes the set of points in R^2 with bounded backward orbits then the set $K(F) = K^+(F) \cap K^-(F)$ is called filled Julia set .

Definition (2-8)[4]

Let F: $R^2 \longrightarrow R^2$ be a map. Let $J^{\pm}(F) = \partial K^{\pm}(F)$,and $J(F) = J^+(F) \cap J^-(F)$ then $J^{\pm}(F)$ is called the forward /backward Julia set and J(F) is the Julia set of F.

Theorem (2-9)[4]

Let F be a hyperbolic regular polynomial automorphism of C^n with $|\det DF| \le 1$.Then Ω(F) = J(F)$\cup$ {$\alpha_1, \alpha_2, \ldots, \alpha_m$}, where the α_i are the attracting periodic points of F.

Theorem (2-10)

Let $a < a_0(b)$, $b > 0$, the Henon map $H_{a,b}$ has no non wandering point , that is $\Omega(H_{a,b}) = \phi$.

Proof: since $b < 1$ and from definition (1-1) $|\det DF| = \left|\det\begin{pmatrix} -2x & -b \\ 1 & 0 \end{pmatrix}\right| = |b| < 1$,

for $\begin{pmatrix} x \\ y \end{pmatrix} \in R^2$ we have three cases :

case 1 : If $\begin{pmatrix} x \\ y \end{pmatrix} \in S_1$, from lemma (2-4) part (ii) , $\langle x_n \rangle$ is strictly decreasing sequence along $H_{a,b}$-orbits and $\langle x_n \rangle$ is not bounded below that is $\langle x_n \rangle \longrightarrow -\infty$ as $n \longrightarrow \infty$ for all $\begin{pmatrix} x \\ y \end{pmatrix}$ in S_1 ,that is there is no element in S_1 such that has bounded forward orbits ,so by definition (2-7) $K^+(H_{a,b}) = \phi$ and by definition (2-8) $J^+(H_{a,b}) = \partial(\phi)$ for all $\begin{pmatrix} x \\ y \end{pmatrix}$ in S_1,since $\partial(\phi) = \phi$,we have $J^+(H_{a,b}) = \phi$ hence by definition (2-8) $J(H_{a,b}) = \phi \cap J^-(H_{a,b}) = \phi$.

Case 2:If $\begin{pmatrix} x \\ y \end{pmatrix} \in S_2$, from lemma (2-4) part (iv) the sequence $\langle y_{-n} \rangle$ is strictly decreasing sequence along $H_{a,b}^{-1}$-orbit and it is not bounded below , that is $\langle y_{-n} \rangle \longrightarrow -\infty$ as $n \longrightarrow \infty$ for all $\begin{pmatrix} x \\ y \end{pmatrix}$ in S_2 ,that is there is no element in S_2 such that has bounded backward orbit so by definition (2-7) $K^-(H_{a,b}) = \phi$ and by definition (2-8) $J^-(H_{a,b}) = \partial(\phi)$ for all $\begin{pmatrix} x \\ y \end{pmatrix}$ in S_2 ,we have $\partial(\phi) = \phi$ so $J^-(H_{a,b}) = \phi$, hence by definition (2-8) we get $J(H_{a,b}) = J^+(H_{a,b}) \cap \phi = \phi$.

Case 3: If $\begin{pmatrix} x \\ y \end{pmatrix} \in S_3$,by lemma (2-4) part (i) $H_{a,b}(S_3) \subset S_1^{\,0}$,hence $H_{a,b}(S_3) \cap S_3 \subset S_3 \cap S_1^{\,0}$, since $S_3 \cap S_1^{\,0} = \phi$, we have $H_{a,b}(S_3) \cap S_3 = \phi$ $H_{a,b}(S_3) \subset S_1^{\,0}$, so S_3 maps into S_1, hence by case 1 ,we have $J^+(H_{a,b}) = \phi$. So from definition (2-8), $J(H_{a,b}) = \phi \cap J^+(H_{a,b}) = \phi$ .Now for all $\begin{pmatrix} x \\ y \end{pmatrix}$ in R^2, we get $J(H_{a,b}) = \phi$,by theorem (2-5), $H_{a,b}$ has no periodic point for any period ,hence by theorem (2-9) $\Omega(H_{a,b}) = \phi$. □

Referencess

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