SOLUTION OF PARABOLA EQUATION BY USING REGULAR ,BOUNDARY AND CORNER FUNCTIONS

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Abstract:-

we solve convergent sequence by using the parabola equation which have a small positive parameter ξ and find a unique solution for a given convergent sequence.

1-Introduction :

- Differential equations a very important tool for solving many phenomenas .There are many authers which studied the applied mathematics as physical mathematics .Levenshtam [4] studied the ordinary differenatiol equations of the first order and the first degree .Dieudonne [1] studied the ordinary equation of the second degree with initial and boundary conditions .Techanoff and Samarcki [5] studied only the partial differenatiol equations of the first order and the first degree with boundary conditions .Levenshtam [3] studied the parabola equation (partial differential equation of the second order and the first degree) .They put initial and boundary conditions to solve this equation .They used converge series from uniform function with two boundary functions .They proved the existence and uniquness of the solution .

One of the mathematical branch which take care by studying phenomena's which comes from the the environment we live in. Several mathematical models are formed for many researchers to solve these phenomena's and really most of these models have a high ability to study see (Kreysig [2] and Smith [6]) .

Some effects on the accuracy of the solution may be small and some researchers don’t take care to study it and about the importance of these effect may be have an effect on the result, of the phenomena's and inversely, on the other side some researchers are emphatics that these effects must be studied and one of them is (Dieudonne J.[1]) , we also like him.

In this work ,we devlope the reserch of [5] ,we study the converge series which is uniform function ,two boundary functions and two corner functions (see section 4 equation (6)).We prove the converge function is unique solution of parabola equation .

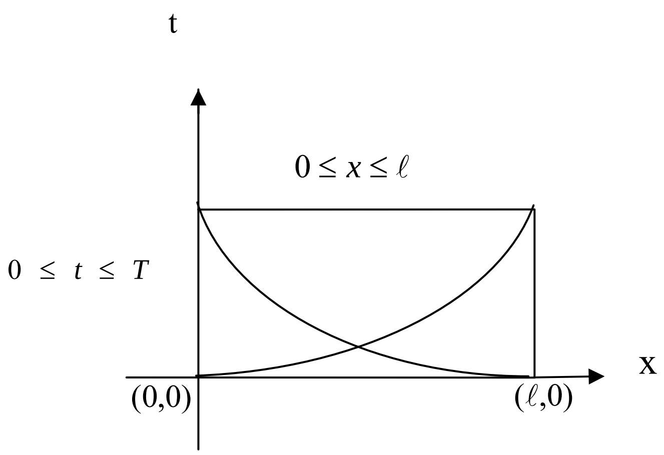
2-Main problem

In this research, we study the following problem :-

$$\xi^2\left(\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}\right)=\sum_{i=1}^{\infty}\xi^i f_i(x,t,u) \quad , 0<x<\ell \ , 0<t<\infty \ \ldots\ldots\ldots (1)$$

$$u(x,0,\xi)=\varphi(x) \quad , \quad \frac{\partial u}{\partial x}(0,t,\xi)=\frac{\partial u}{\partial x}(\ell,t,\xi) \qquad \ldots\ldots\ldots\ldots (2)$$

where u is an arbitrary vector and ξ a very small positive parameter .We will discuss the problem in the region $\Re=\{0\le x\le \ell\}\times\{0\le t\le \mathrm{T}\}$
We will prove that the analytic convergent series
$u(x,t,\xi)=\sum_{i=1}^{\infty}\xi^i u_i(x,t)$ is a solution for the problem (1) with boundary conditions and initial condition in (2).In order to prove the existence of a solution for (1) and (2)., we find an estimation for the convergent series. In other meaning, for construct like this convergence and for more accurate this contains two functions:
-The first is the function, on the boundary of $\Re$ when $t=0, x=0, x=\ell$.
-The corner function at the points (0,0), (0,ℓ) in the region $\Re$.



3-Some Special conditions to Solve The Main Problem

We can find the solution of our main problem (1),and by using the following conditions :-

I– The functions $f(x,t,u,\xi)=\sum_{i=1}^{\infty}\xi^i f_i(x,t,u)$ and $\varphi(x)$ which are (n+2) differentiable **to** construct a convergent series of order n and satisfy (2) at (0, 0), (ℓ, 0) $\ni$ $\varphi(0)=\varphi(\ell)=0$.

From (1) and when $\xi=0$ we get the following equation

f(x, t,$\bar{u}$,0) = 0 ……… …. (3)

II – Equation(3) at the region $\Re$ has an arbitrary solution assume it $\bar{u}_0 = \bar{u}_0(x,t)$

III – When x is a parameter $(0 \le x \le \ell)$ then:-

$$\frac{dB_0}{d\tau} = f(x,0,\bar{u}_0(x,0) + B_0,0) \quad , \tau > 0 \qquad \ldots\ldots\ldots\ldots \textbf{(4)}$$

Iν – By using of the conditions in (2), the solution of equation (4) becomes

$$B_0(x,0) = \varphi(x) - \bar{u}_0(x,0) \qquad \ldots\ldots\ldots\ldots \textbf{(5)}$$

4-

The convergent series which is analytic with respect to ξ to solve (1) and (2) it give as follow:-

$$u(x,t,\xi) = \bar{u}(x,t,\xi) + B(x,\tau,\xi) + Q(\zeta,t,\xi) + Q^*(\zeta^*,t,\xi) + P(\zeta,\tau,\xi) + p^*(\zeta^*,\tau,\xi).$$
(6)

Such that $\tau = \frac{t}{\xi^2}, \zeta = \frac{x}{\xi}, \zeta^* = \frac{\ell - x}{\xi}$

Where $\bar{u}$ is the regular function Q, Q^*, B boundary functions and p, p* the corner functions which are series raised to powers with respect to ξ, for example

$$\bar{u}(x,t,\xi) = \sum_{i=0}^{\infty} \xi^i \bar{u}_i(x,t).$$

To find the coefficients of these series ,we put (6) in (1), (2) and at the function f(x, t, u, ξ) as follows:-

$$f(x,t,u,\xi) = \sum_{i=0}^{\infty} \xi^i f_i(x,t,\bar{u}_i(x,t) + B_i(x,\tau) + Q_i(\zeta,t) + Q^*(\zeta^*,t) + P_i(\zeta,\tau) + P_i^*(\zeta^*,\tau)$$

we can write it as below

$f = \bar{f} + Bf + Qf + Q^* f + Pf + P^* f$ such that

$\bar{f}(x,t,\xi) = f(x,t,\bar{u}(x,t,\xi),\xi)$;

$Bf(x,\tau,\xi) = f(x,\xi^2\tau,\bar{u}(x,\xi^2\tau,\xi) + B(x,\tau,\xi),\xi) - \bar{f}(x,\xi^2\tau,\xi)$;

$Qf(\zeta,t,\xi) = f(\xi\zeta,t,\bar{u}(\xi\zeta,t,\xi) + Q(\zeta,t,\xi),\xi) - \bar{f}(\xi\zeta,t,\xi)$;

$Pf(\zeta,t,\xi) = f(\xi\zeta,\xi^2\tau,\bar{u}(\xi\zeta,\xi^2\tau,\xi) + B(\xi\zeta,\tau,\xi) + Q(\zeta,\xi^2\tau,\xi) + P(\zeta,\tau,\xi),\xi)$
$- Bf(\xi\zeta,\tau,\xi) - Qf(\zeta,\xi^2\tau,\xi) + \bar{f}(\zeta,\xi^2\tau,\xi)$;

By the same way , we can define $Q^* f, P^* f$.

By the standard procedures for the analysis of series with respect to ξ powers and by putting the coefficients for the equal powers with respect to ξ ,we get an equation for any convergent series, for example when ξ^0:

(a) the function u_0 has The following equation:-

$f(x,t,\bar{u}_0,0)=0$

Which can be found by the correspondence with the equation (3) ,thus we get $\bar{u}_0=\bar{u}_0(x,t)$ and for the functions $\bar{u}_i(x,t)$, $i \geq 1$.

We have the following linear equations $\bar{f}_u(x,t)\bar{u}_i(x,t)=f_i(x,t)$ where $f_i(x,t)$ represented by $\bar{u}_j(x,t)$, j< i and from this

$\bar{u}_i(x,t)=f_u^{-1}(x,t)f_i(x,t)$.

(b) Equation of the function $B_0(x,\tau)$ is

$\frac{\partial B_0}{\partial \tau}=B_0 f \equiv f(x,0,\bar{u}_0(x,0)+B_0(x,\tau)-f(x,0,\bar{u}_0(x,0),0))$,this correspond with the equation (4) added to the boundary condition in (5) from the conditions (I – IV), the function $B_0(x,\tau)$ have the following expontioal estimation (see [1] page 57).

$$\|B_0(x,\tau)\| \leq C\exp(-b\tau) \quad , \quad 0 \leq x \leq \ell \quad ,\tau > 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(7)}$$

Where b>0, c>0 are arbitrary constants .The functions $B_i(x,\tau)$ when $i \geq 1$ is linear with the formula

$$\frac{\partial B_i}{\partial \tau}=f_u(x,0,\bar{u}_0(x,0)+B_0(x,\tau),0)B_j+\pi_i(x,\tau)$$

We define $B_j(x,0)=-\bar{u}_i(x,0)$ and $\pi_i(x,\tau)$ represented by $B_j(x,\tau)$ where $j<i$.

The solution for these linear equations which possesses the expontioal estimation is as in (7).

5- The Main Resaults

We find the function $Q_0(\zeta,t)$,where t is as a parameter from the following equation

$$-\frac{\partial^2 \varphi_0}{\partial \zeta^2}=Q_0 f \equiv f(0,t,\bar{u}_0(0,t)+Q_0(\zeta,t),0)-f(0,t,\bar{u}_0(0,t),0)$$

From the equation (2) for $Q_i(\zeta,t)$,we get the two conditions

$$\frac{\partial \varphi_0}{\partial \zeta}(0,t)=0 \ , \ \frac{\partial \varphi_i}{\partial \zeta}(0,t)=\frac{-\partial \bar{u}_{i-1}}{\partial x} \quad , \ i \geq 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(8)}$$

The boundary condition in (2) gives $Q_i(\xi,t)\to 0$ when $\zeta \to \infty$, $i \geq 0$ so $Q_0(\zeta,T)\equiv 0$ while the functions $Q_i(\zeta,t)$, $i \geq 1$ are defined from the linear equations ,(ζ constant coefficient)

$$-\frac{\partial^2 Q_i}{\partial \zeta^2}=\bar{f}_u(0,t)Q_i+q_i(\zeta,t) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(9)}$$

We can find $q_i(\zeta,t)$ through the functions $Q_j(\zeta,t)$, $j<i$ The solutions of (8) and (9) and from condition III with the boundary condition ,we get the single values which have the following expontioal estimation

$\|Q_i(\zeta,t)\| \le C\exp(-b\zeta)$, $\zeta \ge 0$, $0 \le t \le T$ ....(10)

The function $P(\zeta,\tau,\xi)$ has a role to make the function B active with the boundary conditions and the function Q with its initial condition also, the linear differential equation for $P_0(\zeta,\tau)$ is

$$\frac{\partial p_0}{\partial \tau}-\frac{\partial^2 p_0}{\partial \zeta^2}=p_0 f \equiv f(0,0,\bar{u}_0(0,0)+B_0(0,\tau)+P_0(\zeta,\tau),0)-f(0,0,\bar{u}_0)+B_0(0,\tau),0),$$

$\zeta>0$, $\tau>0$

By substitution (6) in the conditions of (2) and for the function $P_i(\zeta,\tau)$ we get

$$p_0(\zeta,0)=0, \frac{\partial p_0}{\partial \zeta}(0,\tau)=0 \text{ .... (11)}$$

$$p_i(\zeta,0)=-Q_i(\zeta,0), \frac{\partial p_i}{\partial \zeta}(0,\tau)=-\frac{\partial B_{i-1}}{\partial x}(0,t), i\ge 1 \text{ .... (12)}$$

From the conditions (11) and (12), the function p is boundary function for both variables ,that is $p_i(\zeta,\tau)\to 0$ when $\sqrt{\zeta^2+\tau^2}\to\infty$............ (13)

At it $p_0(\zeta,\tau)\equiv 0$,while the functions $p_i(\zeta,\tau)$ can be defined from the linear parabola equations $\frac{\partial p_i}{\partial \tau}-\frac{\partial^2 p_i}{\partial \zeta^2}-A(\tau)p_i \equiv H_i(\zeta,\tau)$ .... (14)

where $A(\tau)=P_u(0,0,\bar{u}_0(0,0)+B_0(0,\tau),0)$.

The function $H_i(\zeta,\tau)$represents the functions $Q_i, B_j, j\le i$ and $p_j, i>j$. The solutions of (12) and (14) can be defined by (Techanoff , Samarcki [4])

$$p_i(\zeta,\tau)=g(\zeta,\tau)+\int_0^\tau d\tau_0\int_0^\infty G(\zeta,\tau,\zeta_0,\tau_0)h(\zeta_0,\tau_0)d\zeta_0 \ ;$$

here $g(\zeta,\tau)$ is an arbitrary function which is differentiable (smooth) satisfy the conditions (12) and (13)

$$h(\zeta,\tau)=H_i(\zeta,\tau)-\frac{\partial g}{\partial \tau}+\frac{\partial^2 g}{\partial \zeta^2}+A(\tau)g \quad ,$$

$G(\zeta,\tau,\zeta_0,\tau_0)$ _ (Greena function),

$$G(\zeta,\tau,\zeta_0,\tau_0)=\phi(\tau)\phi^{-1}(\tau_0)\frac{1}{2\sqrt{\pi a(\tau-\tau_0)}}\left\{\exp\left[\frac{-(\zeta-\zeta_0)^2}{4a(\tau-\tau_0)}\right]+\exp\left[\frac{-(\zeta+\zeta_0)^2}{4a(\tau-\tau_0)}\right]\right\}$$

, where $\phi(\tau)$ is fundamental matrix have the following expontioal estimation [1]:

$\left\|\phi(\tau)\phi^{-1}(\tau_0)\right\| \le C\exp[-b(\tau-\tau_0)]$, thus the expontioal estimation for the function p becomes :-

$\left\|p_i(\zeta,\tau)\right\| \le C\exp[-C(\zeta+\tau)], \zeta \ge 0, \tau \ge 0$ .... (15)

To find the estimation of the functions P^*, Q^* by same method used for P and Q and having the same expontioal estimation as in (10) and (15) U_n represent to the convergent series of order n for the series(6)

$$U_n = \sum_{i=0}^{n} \xi^i \left[\overline{u}_i(x,t) + B_i(x,\tau) + Q_i(\zeta,t) + Q_i^*\left(\zeta^*,t\right) + P_i(\zeta,t) + P_i^*\left(\zeta^*,\tau\right)\right]$$

Therefore the following statement is valid.

Theorem (5-1):

If the conditions (I – IV) are valid then $u(x,t,\xi),(\xi$ small parameter) have a unique solution to the problems (1) ,(2) and for the uniform convergent series U_n in R converges to the estimation $O\left(\xi^{n+1}\right)$ i.e.

$$\max_R \left\|u - U_n\right\| = O\left(\xi^{n+1}\right)$$

Proof:-

Let w= u – U ;

$U = U_n + \xi^{n+1}\left(Q_{n+1} + Q_{n+1}^* + P_{n+1} + P_{n+1}^*\right)$, where U_n the partial convergent series defined in (6). Thus we have the following equation:-

$$\xi^2\left(\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2}\right) - f_u(x,t,\xi)w = h(w,x,t,\xi) \text{ ,............................ (17)}$$

where $f_u(x,t,\xi) = f_u\left(\overline{u}_0(x,t) + B_0(x,t/\zeta), x,t,0\right)$,

$$h(w,x,t,\xi) = f(U+w,x,t,\xi) - \xi^2\left(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}\right) - f_u(x,t,\xi)w$$

The function $h(w,x,t,\xi)$ have the following two properties:-

1- when w = 0 ,it is clear that the equation

$$h(0,x,t,\xi) = f(u,x,t,\xi) - \xi^2\left(\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2}\right) = O\left(\xi^{n+1}\right)$$

2- If $\left\|w_i(x,t,\xi)\right\| \le c_1\xi, i = 1,2$ then there exist $c_2 > 0, \xi_0 > 0$ such that $0 < \xi \le \xi_0$ satisfy the following inequality

$$\sup_R \left\|h(w_2,x,t,\xi) - h(w_1,x,t,\xi)\right\| \le C_2 \sup_R \left\|w_2 - w_1\right\| .$$

By using (Greena matrix), equation (1) with conditions (2) ,(3) becomes

$$w(x,t,\xi) = \int_0^t \int_0^l G(x,t,x_0,t_0,\zeta)h(w(x_0,t_0,\zeta),x_0,t_0,\zeta)d\,x_0\,d\,t_0 \equiv L(w,x,t,\zeta)$$

......(18)

From condition III, Greena matrix have the estimation (Techanoff , Samarcki [4]):

$$//G(x,t,x_0,t_0,\xi)// \le \frac{c}{\xi^2}\frac{1}{\sqrt{t-t_0}}\exp[-c\frac{t-t_0}{\xi^2}]\exp[-c\frac{(x-x_0)^2}{t-t_0}].$$

By applying the theorem of the convergence of sequences for the equation (18) ,we get $w_0 = 0, w_{i+1} = L(w_i, x, t, \xi), i = 1,2,.....$

From above, we conclued that $w(,x,t,\xi)$ is a unique solution for (2) when ξ is a small parameter and have the estimation

$$\max_R \|w(x,t,\xi)\| = O(\xi^{n+1})$$, thus we get

$$\max_R \|u - U_n\| = O(\xi^{n+1}) .$$

References:-

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الخلاصة:-

في هذا البحث نحل متتابعة متقاربة باستخدام معادلة قطع مكافئ والتي تمتلك معلمة صغيرة وإيجاد حل وحيد لتلك المتتابعة المتقاربة.