The dynamics of Henon Map $H_{a,b} \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} a - by - x^2 \\ x \end{array} \right)$ Where $a = 0, -1 < b < 0$

Iftichar Mudar Talb and Kawa Ahmad Hassan

Department of Mathematics, College of Education, Babylon University.

Abstract

We prove the Henon map $H_{a,b}$, $-1 < b < 0$, has no periodic point for any period in the plane. $H_{a,b}$ has attracting fixed point $\left( \begin{array}{c} 0 \\ 0 \end{array} \right)$ and a saddle fixed point. There exists an open set about $\left( \begin{array}{c} 0 \\ 0 \end{array} \right)$ in which all points tend to $\left( \begin{array}{c} 0 \\ 0 \end{array} \right)$ under forward iteration of $H_{a,b}$.

1-Introduction

About 30 years ago the French astronomer –mathematician Michel-Henon was searched for a simple two-dimensional map possessing special properties of more complicated systems. The result was a family of maps denoted by $H_{a,b}$ given by $H_{a,b} \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} 1 - ax^2 + y \\ bx \end{array} \right)$ where $a, b$ are real numbers. These maps defined in above are called Henon maps [3]. Henon map $H_{a,b} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. The importance of Henon's map for the dynamics of all polynomial diffeomorphisms was recognized by S. Friedland and J. Milnor, who proved that every quadratic polynomial diffeomorphism is conjugate to a Henon map or to an elementary maps have trivial dynamics. Henon map is normal form for all quadratic polynomial diffeomorphism with non-trivial dynamic.[4]. This work we care this form of Henon map $H_{a,b} \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} a - by - x^2 \\ x \end{array} \right)$.

To determine the Henon map $H_{a,b}$, $-1 < b < 0$, has no periodic point for any period in the plane by dividing the plane to some region which are shown in section four.
we conclude that there exists an open set about 0\(0\) in which all points tend to 0\(0\) under forward iteration of H. We found these regions \(Q_1, Q_{2,1}, Q_{2,2}, Q_3, Q_4\) and \(R_1, R_2, R_3, R_4, R_5, R_6, R_7\), we prove there are no periodic point for Henon map in the plane. Also, we prove that the norm of iteration of Henon map \(H_{0,b}\) tends to infinity for some regions shown in proposition (4-2) and (4-4).

In section two, we recall some necessary definitions and theorems. In section three, we introduce forward and backward iteration of Henon map \(H_{a,b}\), where \(a = 0, -1 < b < 0\). In section four, we study the type of fixed point of Henon map \(H\) with a basin of attraction of one of fixed points. In section five we study non existence of periodic point of Henon map in \(R^2\) by finding some region in \(R^2\). At the last, we introduce forward and backward iteration of Henon map \(H_{a,b}\), where \(a = 0, b > 0\). we conclude that there exists an open set about 0\(0\) in which all points tend to 0\(0\) under forward iteration of H., we prove that the norm of iteration of Henon map \(H_{0,b}\) tends to infinity for some regions.

2-Preliminaries

The purpose of this section is to introduce some definitions and theorems necessary for this research, we recall some fundamental definitions and necessary theorems. Let \(A\) be an \(n \times n\) matrix. The real number \(\lambda\) is called an eigen value of \(A\) if there exists a non zero vector \(X\) in \(R^n\) such that \(AX = \lambda X\) \(\text{(2.1)}\)

Every non zero vector \(X\) satisfying \(\text{(2.1)}\) is called an eigen vector of \(A\) associated with the eigen value \(\lambda\). In our work, we write the map \(F: V \rightarrow \mathbb{R}^2\) where \(V \subseteq \mathbb{R}^2\), such that \(F = \begin{bmatrix} f(x) \\ g(y) \end{bmatrix} ; \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2\). Also, the forward orbit of a vector \(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}\) is the set of points \(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, F\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, F^2\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, F^3\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \ldots\) and denoted by \(O^+(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix})\). If \(F\) a homeomorphism, we may define the full orbit of \(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}\), \(O(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix})\), as the set of points \(F^n(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix})\) for \(n \in \mathbb{Z}\), and the backward orbit of \(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}\), \(O^-(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix})\), as the set of points \(F^{-n}(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix})\), \(F^{-2}(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix})\), \(F^{-3}(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix})\), \ldots, where we have \(F^n(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}) = F^{n-1}(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}) = \ldots = \begin{bmatrix} x_n \\ y_n \end{bmatrix}\).
Definition (2-1)[6] Any pair \( \begin{pmatrix} p \\ q \end{pmatrix} \) for which 
\[ f \begin{pmatrix} p \\ q \end{pmatrix} = p, \quad g \begin{pmatrix} p \\ q \end{pmatrix} = q \] (2.2)
is called a fixed point of the two dimensional dynamical system.

Definition (2-2)[1] Let \( F: \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a map, \( x_0 \in \mathbb{R}^n \). The point \( x_0 \) is a periodic point of period \( m \) if
\[ F^m(x_0) = x_0. \] The least positive integer \( m \) for which \( F^m(x_0) = x_0 \) is called the prime period of \( x_0 \).

Example (2-3) Let \( F: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be given by 
\[ F = \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} y \\ x \end{pmatrix} \], then \( \begin{pmatrix} 1 \\ 2 \end{pmatrix} \) is a periodic point of period 2 of \( F \).

Definition (2-4)[2] Let \( \begin{pmatrix} p \\ q \end{pmatrix} \) be a fixed point of \( F \), then \( \begin{pmatrix} p \\ q \end{pmatrix} \) is attracting fixed point if and only if there is a disk centered at \( \begin{pmatrix} p \\ q \end{pmatrix} \) such that \( F^n(\begin{pmatrix} x \\ y \end{pmatrix}) \rightarrow \begin{pmatrix} p \\ q \end{pmatrix} \) for every \( \begin{pmatrix} x \\ y \end{pmatrix} \) in the disk as \( n \rightarrow \infty \). By contrast \( \begin{pmatrix} p \\ q \end{pmatrix} \) is repelling fixed point if and only if there is a disk centered at \( \begin{pmatrix} p \\ q \end{pmatrix} \) such that
\[ \left\| F \begin{pmatrix} u \\ v \end{pmatrix} - F \begin{pmatrix} p \\ q \end{pmatrix} \right\| > \left\| \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} p \\ q \end{pmatrix} \right\| \] for every \( \begin{pmatrix} u \\ v \end{pmatrix} \) in the disk for which \( \begin{pmatrix} u \\ v \end{pmatrix} \neq \begin{pmatrix} p \\ q \end{pmatrix} \).

Theorem (2-5)[2] Let \( \begin{pmatrix} p \\ q \end{pmatrix} \) be a fixed point of \( F \). Assume that \( DF \begin{pmatrix} p \\ q \end{pmatrix} \) exists, with eigen values \( \lambda_1, \lambda_2 \), then:

1. \( \begin{pmatrix} p \\ q \end{pmatrix} \) is an attracting fixed point, if \( \lambda_1, \lambda_2 \) are less than one in absolute value.
2. \( \begin{pmatrix} p \\ q \end{pmatrix} \) is repelling fixed point, if \( \lambda_1, \lambda_2 \) are greater than one in absolute value.
3. \( \begin{pmatrix} p \\ q \end{pmatrix} \) is saddle point, if one of \( \lambda_1, \lambda_2 \) is larger and the other is less than one in absolute value.

Example (2-6)[2] Let \( F \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ a \sin(x) - y \end{pmatrix} \) then \( \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) is an attracting fixed point if \(-\frac{1}{4} < a < 0\), \( \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) is a repelling fixed point if \( a > 2\), \( \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) is saddle fixed point if \( 0 < a < 2\).

Definition (2-7)[1]
Let \( V,S \subseteq \mathbb{R}^2 \), \( F: V \rightarrow V \) and \( G:S \rightarrow S \) be two maps. Then \( F \) and \( G \) are said to be topologically conjugate if there exist a homeomorphism \( H: V \rightarrow S \) such that \( H \circ F = G \circ H \). The homeomorphism \( H \) is called a topological conjugacy.
Example (2-8)[6]

Let \( Q(x) = 4x(1-x) \) for \( 0 \leq x \leq 1 \) and \( T(x) = \begin{cases} 
2x & \text{for } 0 \leq x \leq \frac{1}{2} \\
2(1-x) & \text{for } \frac{1}{2} < x \leq 1
\end{cases} \).

\( Q \) is topologically conjugate to \( T(x) \). Where topological conjugacy is \( \sin^2 \frac{x}{2} \).

Definition (2-9)[3]

Let \( \begin{pmatrix} p \\ q \end{pmatrix} \) be a fixed point of \( F \). The **basin of attraction** of \( \begin{pmatrix} p \\ q \end{pmatrix} \) consists of all \( \begin{pmatrix} x \\ y \end{pmatrix} \) such that \( F^{n}\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} p \\ q \end{pmatrix} \) as \( n \to \infty \).

Theorem (2-10)[1]

Suppose that \( F \) has an attracting fixed point at \( \begin{pmatrix} p \\ q \end{pmatrix} \). Then there is an open set about \( \begin{pmatrix} p \\ q \end{pmatrix} \), in which all points tend to \( \begin{pmatrix} p \\ q \end{pmatrix} \) under forward iteration of \( F \).

Proposition (2.11)[1],[5]

Let \( H: \mathbb{R}^2 \to \mathbb{R}^2 \) be a Henon map and \( b \) be any fixed real number. Then

1. \( J_{H_{a,b}}(x,y) = b \), \( \forall x, y \in \mathbb{R} \).

2. If \( x^2 - b \geq 0 \), then the eigen values of \( D_{H_{a,b}}(x,y) \) are the real numbers \( x \pm \sqrt{x^2 - b} \).

3. If \( b \neq 0 \), \( H_{a,b} \) is one-to-one map

4. The Henon map \( H_{a,b} \) is \( C^\infty \).

5. If \( b \neq 0 \) then \( H_{a,b} \) has an inverse.

6. If \( b \neq 0 \) \( H_{a,b} \) is diffeomorphism.

Proposition (2.12)[1]

For each value of \( b \) in \( \mathbb{R} \), there exists \( b_0 \in \mathbb{R} \setminus \{0\} \) such that, if \( a > b_0 \) the Henon map has two fixed points.
3-Type of fixed points with basin of attraction

Our goal of this section is to determine type of fixed points of $H_{a,b}$ with the basin of attraction of a fixed point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. We will use maximum norm, where the maximum norm of $\begin{pmatrix} x \\ y \end{pmatrix}$ is $\max\{|x|, |y|\}$.

**Proposition (3-1)**

If $-1 < b < 0$, then $H_{a,b}$ has attracting fixed point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and a saddle fixed point $\begin{pmatrix} -(b+1) \\ -(b+1) \end{pmatrix}$.

Proof: Since $a = 0$ and $-1 < b < 0$. By proposition (2.12) $P_{+} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $P = \begin{pmatrix} -(b+1) \\ -(b+1) \end{pmatrix}$ are fixed points for $H_{a,b}$. By proposition (2.11) $\lambda_{1,2} = \pm \sqrt{-b}$ are eigen values of $DH_{a,b} \begin{pmatrix} x \\ y \end{pmatrix}$ at $P_{+}$. Since $-1 < b < 0$ then $0 < \sqrt{-b} < 1$ and $|\lambda_1| = |\lambda_2| < 1$, so $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is attracting fixed point. To show that $P$ is a saddle fixed point. From proposition (2.11), we have two eigen values, $\lambda_1 = -(b+1) + \sqrt{(b+1)^2 - b}$, $\lambda_2 = -(b+1) - \sqrt{(b+1)^2 - b}$

(3.1)

Since $b^2 + b + 1 < (b+2)^2$ thus $\sqrt{b^2 + b + 1} < \sqrt{(b+2)^2} = b+2$ hence $-(b+1) + \sqrt{b^2 + b + 1} < 1$ (3.2)

On the other hand, since $\sqrt{b^2 + b + 1} = \sqrt{(b+1)^2 - b} > \sqrt{(b+1)^2} = |b+1| = b+1$

so $-(b+1) + \sqrt{b^2 + b + 1} > 0$, hence by (3.2) $|-(b+1) + \sqrt{(b+1)^2 - b}| = |\lambda_2| < 1$. (3.3)

For $\lambda_1$ since $-1 < b < 0$ we have $\sqrt{b^2 + b + 1} > |b| = -b$. (3.4)

By adding $(b+1)$ for both sides of (3.4), we get $(b+1) + \sqrt{b^2 + b + 1} > 1$

then $|b+1 + \sqrt{(b+1)^2 - b}| = |\lambda_1| > 1$. (3.5) Hence from (3.4), (3.5) and theorem (2-5), $P$ is a saddle fixed point. □
Remark (3-2) In view of theorem (2-10), there exists an open set about \( \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) in which all points tend to \( \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) under forward iteration of \( H_{0,b} \). The next theorem shows this open set.

Theorem (3-3) Suppose \( S = \{ \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbb{R}, |x| \leq 1 - |b|, |y| \leq 1 - |b|, |b| < 1 \} \) then for all \( \begin{pmatrix} x \\ y \end{pmatrix} \in S^* \), \( H_{0,b} \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) as \( n \rightarrow \infty \).

Proof: We claim that \( H_{0,b} (S^*) \subset S^* \), to show this, let \( \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \) be any element in \( H_{0,b} (S^*) \) then there is \( \begin{pmatrix} x \\ y \end{pmatrix} \) in \( S^* \) such that \( H_{0,b} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \). From this \( |y_1| = |x| < 1 - |b| \) (3.6)

and \(|x_i| = |by + x^2| \leq |b||x| + |x|^2 \leq |b|(1 - |b|) + (1 - |b|)^2 = |b| - |b|^2 + 1 - 2|b| + |b|^2 = 1 - |b| \) (3.7)

Now from (3.6), (3.7) \( \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \in S^* \). Let \( r = \begin{pmatrix} x \\ y \end{pmatrix} \), so \( |x| \leq r, |y| \leq r \) and since \( |x| \leq 1 - |b|, |y| \leq 1 - |b| \), then \( r < 1 - |b| \).

We denote \( H_{0,b} \begin{pmatrix} x \\ y \end{pmatrix} \) for \( \begin{pmatrix} x \\ y \end{pmatrix} \) in \( S^* \), by

\[
\begin{pmatrix} x_n \\ y_n \end{pmatrix}, \text{then } |y_2| - |x_i| = |by + x^2| \leq |b||x| + |x|^2 \leq |b| r + r^2 = r(|b| + r) \tag{3.8}
\]

\[
|x_2| = |by_1 + x_1^2| \leq |b| r + r^2 (|b| + r)^2 \tag{3.9}
\]

Now since \( 0 < |b| + r < 1 \), we have \( 0 < r^2 (|b| + r) < r^2 \), then \( |b| + r (|b| + r)^2 < |b| r + r^2 = r(|b| + r) \), hence by (3.8), (3.9) \( |x_2| < r(|b| + r) \) (3.10)

Now we claim that \( |x_{2n}| < r(|b| + r)^n, |y_{2n}| < r(|b| + r)^n \) (3.11)

We prove it by using mathematical induction from (3.8), (3.10), it is true for \( n = 1 \), suppose it is true for \( k \) then \( |x_{2k}| < r(|b| + r)^k, |y_{2k}| < r(|b| + r)^k \) to show that it is true for \( k + 1, \)

\[
y_{2(k+1)} = -by_{2(k+1)-2} - x_{2(k+1)-2} = -by_{2k} - x_{2k}.
\]

\[
|y_{2(k+1)}| = |by_{2k} + x_{2k}|^2 \leq |b||y_{2k}| + |x_{2k}|^2 \leq |b| r(|b| + r)^k + r^2 (|b| + r)^2 \tag{3.12}
\]
Now \( 0 < (|b| + r) \cdot k < 1 \), so \( r^2 \cdot (|b| + r) \cdot k + |b| \cdot r < |b| \cdot r + r^2 = r(|b| + r) \).

Hence \( r^2 \cdot (|b| + r) \cdot 2^k + |b| \cdot r \cdot (|b| + r) \cdot k < r(|b| + r) \cdot k+1 \) \hspace{1cm} (3.13)

Hence from (3.12), we get \( |y_{2(k+1)}| < r(|b| + r) \cdot k+1 \) \hspace{1cm} (3.14)

Also, we have \( x_{2(k+1)} = -by_{2(k-1)} - x_{2(k+1)}^2 = -by_{2k+1} - x_{2k+1}^2 = -bx_{2k} - y_{2(k+1)} \), since \( x_n = y_{n+1} \)

so \( |x_{2(k+1)}| < |bx_{2k} + y_{2(k+1)}| < |b||x_{2k}| + |y_{2(k+1)}|^2 < |b| \cdot r(|b| + r) \cdot k + r^2(|b| + r) \cdot 2^k \) hence from (3.13) we get that \( |x_{2(k+1)}| < r(|b| + r) \cdot k+1 \), hence it is true for \( k+1 \).

Now \( \lim_{n \to \infty} \|H_{0,b}^{2n}(x,y)\| = \lim_{n \to \infty} |x_{2n}| \quad \text{as} \quad n \to \infty \). \hspace{1cm} (3.15)

If \( \|H_{0,b}^{2n}(x,y)\| = |y_{2n}| \), as the same as (3.13), \( H_{0,b}^{2n}(x,y) \to (0,0) \) as \( n \to \infty \) and since \( H_{0,b}(S^+ \cap S^+) \subset S^+ \) then \( \forall (x,y) \in S^+ \), we have \( H_{0,b}(x,y) \in S^+ \) by (3.15), so \( H_{0,b}^{2n}(H_{0,b}(x,y)) \to (0,0) \) as \( n \to \infty \), so \( H_{0,b}^{2n+1}(x,y) \to (0,0) \) as \( n \to \infty \), hence \( H_{0,b}^{n}(x,y) \to (0,0) \) as \( n \to \infty \), \( \forall (x,y) \in S^+ \). \( \square \)

4- The Periodic Points for Henon Map Where \( a = 0, -1 < b < 0 \)

In this section, we will show that Henon map \( H_{0,b}, -1 < b < 0 \) has no periodic points other than fixed points in the plane. To prove this, we divide the proof to fifteen lemmas, thus we find some regions such that the union of all regions covers the plane. We prove that there is no periodic point in each of them until we get the main purpose. The regions are the following \( Q_1, Q_{2,1}, Q_{2,2}, Q_3, Q_4 \) and in \( Q_3 \) define the following regions \( R_1, R_2, R_3, R_4, R_5, R_6, R_7 \).

\( Q_1 = \{ (x,y) : x \geq 0, y \geq 0 \} \).

\( Q_{2,1} = \{ (x,y) : x < 0, y \geq 0, x \geq -y^2 \} \).

\( Q_{2,2} = \{ (x,y) : x < 0, y > 0, x < -y^2 \} \).
$Q_4 = \{ (x, y) : x \geq 0, y < 0 \}$.

$R_1 = \{ (x, y) : x \geq -y^2 \}$.

$R_2 = \{ (x, y) : x \leq b\sqrt{-y - y^2} \}$.

$R_3 = \{ (x, y) : y < -\frac{x-x^2}{b}, -y^2 - by < x < -y^2, x < y \}$.

$R_4 = \{ (x, y) : y < -\frac{x-x^2}{b}, b\sqrt{-y - y^2} < x \leq -by - y^2 \}$.

$R_5 = \{ (x, y) : y < -\frac{x-x^2}{b}, -y^2 - by < x < -y^2, x \geq y, by + x^2 > 1 \}$.

$R_6 = \{ (x, y) : y < -1-b, x < -y^2, x \geq y, by + x^2 \leq 1, x > -y^2 - by, x > b\sqrt{-y - y^2} \}$.

$R_7 = \{ (x, y) : x < -1-b, y > -1-b, y > x, y \geq \frac{-x-x^2}{b}, -y^2 - by > x > b\sqrt{-y - y^2} \}$.

**Fig(1): Region $Q_1, Q_{2,1}, Q_{2,2}, Q_3, Q_4$**
Lemma (4-1)

$H_{0,b}$ is topologically conjugate to $H_{0,b}^{-1}$.

Proof: Let $G: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a map defined by $G = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} by \\ bx \end{pmatrix}$. Clearly $G$ is continuous and bijective, $G^{-1}$ is continuous, so $G$ is homeomorphism and $G \circ H_{0,b}^{-1} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x - y^2 \end{pmatrix} = \begin{pmatrix} b \frac{-x - y^2}{b^{-1}} \\ by \end{pmatrix} = \begin{pmatrix} -x - y^2 \\ b^{-2} by \end{pmatrix}$.

(4.1)

From (4.1) and (4.2), $G \circ H_{0,b}^{-1} = H_{0,b} \circ G$. Hence by definition (2-7), $H_{0,b}$ is topologically conjugate to $H_{0,b}^{-1}$. □

Lemma (4-2)

There are no periodic points in $Q_1$ other than fixed points.
Proof: Let \((x_0, y_0) \in Q_1\), we define norm \(\left\| \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right\| = \sqrt{x_0^2 + y_0^2}\). From now for simply we refer to \(H_{0,b}\) as \(H\) and \(H^{-1}_{0,b}\) as \(H^{-1}\). We have \(-1 < b < 0\), \(x_{-1} = y_0, y_{-1} = \frac{-1}{b}(x_0 + y_0)\) then \(y_{-1} = \frac{-1}{b}(x_0 + y_0) > \frac{-1}{b}x_0 > x_0\).

We have \(-1 < b < 0\) so \(x_{-1} > x_0\) and since \(x_{-1}, x_0\) are non negative \(y_{-1}^2 > x_0^2\). That is \(y_{-1}^2 + y_{-1}^2 > x_0^2 + y_0^2\).

thus \(y_{-1}^2 + x_{-1}^2 > x_0^2 + y_0^2\) so \(\sqrt{y_{-1}^2 + x_{-1}^2} > \sqrt{x_0^2 + y_0^2}\), hence \(\left\| \begin{pmatrix} x_{-1} \\ y_{-1} \end{pmatrix} \right\| > \left\| \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right\|\), that is \(H^{-1}(x_0) > (x_0)\). (4.4)

Since \(x_{-1} = y_0 \geq 0, y_{-1} = \frac{-1}{b}(x_0 + y_0) \geq 0\), we have \(H^{-1}(Q_1) \subset Q_1\), then (4.4) is true for all \(n \in Z^+\), hence

\[
\left\| H^{-n}(x_0) \right\| > \left\| H^{-n+1}(x_0) \right\| \quad \forall n \in Z^+, \quad (4.5)
\]

so the norms of the points in the orbit are strictly increasing. Hence there are no periodic points in the region \(Q_1\).

**Lemma (4.3)**

There are no periodic points in \(Q_{1,1}\).

**Proof:** Let \((x_0, y_0) \in Q_{1,1}\), since \(H^{-1}(x_0) = \begin{pmatrix} y_0 \\ \frac{-1}{b}(x_0 + y_0) \end{pmatrix}\), \(y_0 > 0\) and \(x_0 \geq -y_0^2\), hence \(x_{-1} > 0, \frac{-1}{b}(x_0 + y_0) \geq 0\), so \(x_{-1} > 0, y_{-1} \geq 0\), \(\begin{pmatrix} x_{-1} \\ y_{-1} \end{pmatrix} \in Q_1\), that means \(Q_{1,1}\) maps into \(Q_1\) then by lemma (4.2) there are no periodic point for \(H\) in \(Q_{1,1}\). □

**Lemma (4.4)**

There are no periodic points of even period in \(Q_{1,2}\).

**Proof:** Let \((x_0, y_0) \in Q_{1,2}\), From (4.3), since \(y_{-n} = \frac{-1}{b}(x_{-(n-1)} + y_{-(n-1)}^2), x_{-n} = y_{-(n-1)}\). Since \(x_0 < 0, y_0 \geq 0\), so \(x_{-1} = y_0 > 0, y_{-1} = \frac{-1}{b}(x_0 + y_0^2) < 0\), hence \(x_{-2} = y_{-1} < 0, y_{-2} = \frac{-1}{b}(x_{-1} + y_{-1}^2) > \frac{-1}{b}x_{-1}\). Since \(-1 < b < 0\),
\[-\frac{1}{b} x_{-1} > x_{-1} = y_0 > 0, \text{ hence } y_{-2} > y_0 > 0.\] In general, \[y_{-n} > \left(\frac{-1}{b} x_{-(n-1)} > x_{-(n-1)} = y_{-(n-2)}\right), \text{ so } \langle y_{-2n} \rangle\] is strictly increasing sequence. \(H^{-2n} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \) in \(Q_{2,1} \cup Q_{2,2}\) then we have two cases.

Case 1: If \(H^{-2n} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_{2,1}\), then by lemma (4-3) there are no periodic points of even period for \(H\).

Case 2: If \(H^{-2n} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_{2,2}\) and since \(\langle y_{-2n} \rangle\) is strictly increasing sequence so there are no periodic points of even period for \(H\). □

**Lemma (4-5)**

There are no periodic points of odd period in \(Q_4\).

Proof: Let \(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_4\) since \(x_{-1} = y_0 < 0\), \(y_{-1} = \frac{-1}{b} (x_0 + y_0^2), x_0 > 0\) so \(y_{-1} = \frac{-1}{b} (x_0 + y_0^2) > \frac{-1}{b} x_0 > x_0\), hence \(y_{-1} > 0\), from (4.3) \(x_{-(n+1)} = y_{-n}\), we have \(x_{-2} > 0\), then

\[y_{-3} = \frac{-1}{b} (x_{-2} + y_{-2}^2) > \frac{-1}{b} x_{-2} > x_{-2} = y_{-1} > 0, \text{ hence } x_{-4} > 0 \text{ by the same way}
\]

\[y_{-5} = \frac{-1}{b} (x_{-4} + y_{-4}^2) > \frac{-1}{b} x_{-4} > x_{-4} = y_{-3} > 0, \text{ hence } y_{-(2n+1)} > y_{-(2n-1)} > \ldots > y_{-5} > y_{-3} > y_{-1} > 0 \] (4.6)

Either \(x_{-2n-1} > 0\) or \(x_{-2n-1} < 0\) so \(H^{-2n-1} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_1\) or \(H^{-2n-1} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_2\).

Case 1: If \(H^{-2n-1} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_1\), by lemma (4-2), there are no periodic point of odd period.

Case 2: If \(H^{-2n-1} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_2\) since \(\langle y_{-2n-1} \rangle\) is strictly increasing sequence there are no periodic points of odd period. □

**Lemma (4-6)**

There are no periodic points of even period in \(Q_4\).

Proof: Suppose there exists \(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\) in \(Q_4\) such that \(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\) is a periodic point of even period. That is there is \(2n\) in \(\mathbb{Z}^+\), \(H^{-2n} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\). (4.7)
Since \( x_n = y_{-n-1} \), \( y_n = -\frac{1}{b} (x_{-(n-1)} + y_{-(n-1)}) \), then either \( H^{-2n} \left( \begin{array}{c} x_0 \\ y_0 \end{array} \right) \in Q_{2,1} \) or \( H^{-2n} \left( \begin{array}{c} x_0 \\ y_0 \end{array} \right) \in Q_1 \) for all \( n \in N \). If \( H^{-2n} \left( \begin{array}{c} x_0 \\ y_0 \end{array} \right) \in Q_{2,1} \), from (4.7) \( H^{-2n} \left( \begin{array}{c} x_0 \\ y_0 \end{array} \right) = H^{-1} \left( \begin{array}{c} x_0 \\ y_0 \end{array} \right) \) so \( H^{-2n} \left( \begin{array}{c} x_0 \\ y_0 \end{array} \right) = H^{-1} \left( \begin{array}{c} x_0 \\ y_0 \end{array} \right) \), hence \( H^{-2n} (H^{-1} \left( \begin{array}{c} x_0 \\ y_0 \end{array} \right)) = H^{-1} \left( \begin{array}{c} x_0 \\ y_0 \end{array} \right) \), \( H^{-1} \left( \begin{array}{c} x_0 \\ y_0 \end{array} \right) \in Q_{2,1} \) so \( Q_{2,1} \) has a periodic point which is contradiction. If \( H^{-2n} \left( \begin{array}{c} x_0 \\ y_0 \end{array} \right) \in Q_{2,2} \), in the same way, \( H^{-2n} (H^{-1} \left( \begin{array}{c} x_0 \\ y_0 \end{array} \right)) = H^{-1} \left( \begin{array}{c} x_0 \\ y_0 \end{array} \right) \), \( H^{-1} \left( \begin{array}{c} x_0 \\ y_0 \end{array} \right) \in Q_1 \), that means there is a periodic point of even period in \( Q_1 \) which is contradiction, so there are no periodic points of even period in \( Q_1 \). □

**Lemma (4-7)**

There are no periodic points of odd period in \( Q_{2,2} \).

Proof: Suppose that there is a periodic point \( \left( \begin{array}{c} x_0 \\ y_0 \end{array} \right) \) of odd period \( m \). That is there exists at least positive integer \( n \) such that \( m = 2n + 1 \), and \( H^{-2n} \left( \begin{array}{c} x_0 \\ y_0 \end{array} \right) = \left( \begin{array}{c} x_0 \\ y_0 \end{array} \right) \), since \( \left( \begin{array}{c} x_0 \\ y_0 \end{array} \right) \in Q_{2,2} \), we have \( x_0 < -y_0^2, x_{-1} = y_0 \geq 0 \), also \( y_{-1} = -\frac{1}{b} (x_0 + y_0^2) < 0 \), hence \( H^{-1} \left( \begin{array}{c} x_0 \\ y_0 \end{array} \right) \in Q_1 \), that is \( H^{-1} (H^{-2n} \left( \begin{array}{c} x_0 \\ y_0 \end{array} \right)) = H^{-1} \left( \begin{array}{c} x_0 \\ y_0 \end{array} \right) \) also \( H^{-2n} (H^{-1} \left( \begin{array}{c} x_0 \\ y_0 \end{array} \right)) = H^{-1} \left( \begin{array}{c} x_0 \\ y_0 \end{array} \right) \), hence \( \left( x_{-1} \right) \) is a periodic point of odd period in \( Q_4 \) which is contradiction by lemma (4-5), hence there are no periodic points of odd period in \( Q_{2,2} \).

**Lemma (4-8)**

There are no periodic points in \( R_1 \).

Proof: Let \( \left( \begin{array}{c} x_0 \\ y_0 \end{array} \right) \) be any element in \( R_1 \), then \( \left( \begin{array}{c} x_0 \\ y_0 \end{array} \right) \in Q_3, x_0 \geq -y_0^2 \), that is \( y_0 < 0 \)

Since \( x_0 < 0 \), \( x_0 \geq -y_0^2 \), so \( x_{-1} = y_0 < 0 \), \( y_{-1} = -\frac{1}{b} (x_0 + y_0^2) \geq 0 \), hence \( \left( \begin{array}{c} x_{-1} \\ y_{-1} \end{array} \right) \in Q_2 \). That is \( H^{-1} \left( \begin{array}{c} x_0 \\ y_0 \end{array} \right) \in Q_2 \) that means \( R_1 \) maps into \( Q_2 \), so by lemma (4-4) and lemma (4-5) there are no periodic points in \( R_1 \). □

**Lemma (4-9)**
There are no periodic points in $R_2$.

Proof: Let $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in R_2$ then $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_1$, $x_0 \leq b\sqrt{-y_0} - y_0^2$, that is $y_0 < 0$, $x_0 < 0$

$x_0 \leq b\sqrt{-y_0} - y_0^2$. Thus $x_0 + y_0^2 \leq b\sqrt{-y_0}$ so $\frac{-1}{b} (x_0 + y_0^2) \leq -\sqrt{-y_0}$, that is

$y_{-1} \leq -\sqrt{-y_0}$. Hence $y_{-1}^2 \geq -y_0$, that is $y_{-1}^2 \geq -x_1$ so $-y_{-1}^2 \leq x_1$, hence $H^{-1} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in R_1$. Hence $R_2$ maps into $R_1$, by lemma (4-8), there are no periodic point in $R_2$. □

Fig(3): The region $R_1, R_2$ where $b = -0.5$

**Lemma (4-10)**

There are no periodic point in $R_3$.

Proof: Let $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in R_1$, so $y_0 < -\frac{x_0 - x_0^2}{b}$, $-x_0^2 - by_0 < x_0 < -y_0^2$, $x_0 < y_0$, we have $x_1 = -by_0 - x_0^2$, $y_1 = x_0$.

since $-x_0 < y_0^2 + by_0$, $x_0^2 > y_0^2$, we get that $y_0^2 + by_0 < x_0^2 + by_0$, so $-x_0 < x_0^2 + by_0$, that is $x_0 > x_1$, since $x_0 = y_1$, we have $y_1 > x_1$. (4.8)

Suppose that $-x_1 \geq y_1^2 + by_1$, that is $-x_1 \geq x_0^2 + bx_0, bx_0 > by_0$, so $-x_1 > x_0^2 + by_0 = -x_1$ which is contradiction, hence $x_1 > -y_1^2 - by_1$. (4.9)

Clearly, since $-by_0 < 0$, then $-by_0 - x_0^2 < -x_0^2$, hence $x_1 < -y_1^2$. (4.10)
To show that $y_i < \frac{-x_i - x_i^2}{b}$, since $x_0 < y_0, b < 0$, we have $-x_0^2 - by_0 > -x_0^2 - bx_0$, so
$x_1 > -x_0^2 - bx_0 = -x_0^2 - by_1$. \hfill (4.11)

From $y_0 < \frac{-x_0 - x_0^2}{b}$, we have $by_0 > -x_0 - x_0^2$, hence $x_0 > x_1$, since $x_0 < 0$, $x_0^2 < x_1^2$, thus $-x_0^2 - by_1 > -x_1^2 - by_1$, then from (4.11), we get that $x_1 > -x_1^2 - by_1$, that is $by_1 > -x_1 - x_1^2$, hence $y_1 < \frac{-x_1 - x_1^2}{b}$. \hfill (4.12)

Now from (4.9), (4.10), (4.11) and (4.12), we can say that $H(R_3) \subset R_3$, so $H^2(R_3) \subset R_3$ and so on $H^n(R_3) \subset H^{n-1}(R_3) \subset R_3$, where $n$ is positive integer hence, $x_n < y_n, y_n < \frac{-x_n - x_n^2}{b}$, that is $by_n > -x_n - x_n^2$, so $x_n > -x_n^2 - by_n = x_{n+1}$, hence $\langle x_n \rangle$ is strictly decreasing sequence in $R_3$, so there are no periodic points in $R_3$. \hfill □

**Figure 4:** The region $R_3$ where $b = 0.5$

**Lemma (4-11)**

There are no periodic point in $R_4$.

Proof: Let \( \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \) be any element in $R_4$, then $y_0 < \frac{-x_0 - x_0^2}{b}$ \hfill (4.13)
and $b\sqrt{-y_0 - y_0^2} < x_0 \leq -by_0 - y_0^2$. \hfill (4.14)
By (3.28) \( by_0 > -x_0 - x_0^2 \) or \( x_0 > -by_0 - x_0^2 = x_1 \), from (3.18) \( y_1 = x_0 \), so \( y_1 > x_1 \)  \( (4.15) \)

Since \( x_i = -by_0 - x_i^2 \), \( -by_0 < 0 \), we have \( x_i < -y_i^2 \).  \( (4.16) \)

Now, to show that \( x_i > -y_i^2 - by_i \), since \( by_0 > -x_0 - x_0^2 \), \( -x_0 \geq by_0 + y_0^2 \), we get

\[
by_0 - x_0 > -x_0 - x_0^2 + y_0^2 + by_0,
\]

so \( -x_0^2 + y_0^2 < 0 \), that is \( \sqrt{x_0^2} > \sqrt{y_0^2} \) hence \( |x_0| > |y_0| \) since \( \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_3, |x_0| = -x_0, |y_0| = -y_0 \), thus \( x_0 < y_0 \), so \( y_1 < y_0 \) since \( b < 0 \), we have \( x_0^2 + by_0 < x_0^2 + by_1 \), hence,

\[-x_i < x_0^2 + by_1.\]

Now since \( x_0 = y_1 \), we get that \( -x_i < y_1^2 + by_1 \) that is \( x_i > -y_1^2 - by_1 \).  \( (4.17) \)

On the other hand, \( x_0 < y_0 \) thus \( -x_0^2 - by_0 > -x_0^2 - bx_0 \) hence we get that \( x_i > -x_0^2 - by_1 \).  \( (4.18) \)

From \( by_0 > -x_0 - x_0^2 \), then \( x_0 > -x_0^2 - by_0 \), so \( x_0 > x_1 \), \( x_0, x_1 \) are negative, so \( x_0^2 < x_1^2 \), thus

\[-x_0^2 - by_1 > -x_1^2 - by_1, \]

hence from (4.18) \( x_i > -x_1^2 - by_1 \), that is \( y_1 < \frac{-x_i - x_1^2}{b} \)  \( (4.19) \)

Now from (4.15), (4.16), (4.17) and (4.19), we get that \( \begin{pmatrix} x_i \\ y_1 \end{pmatrix} \in R_3 \), so \( R_4 \) maps into \( R_3 \), then by lemma (4-10) there are no periodic points in \( R_4 \). \( \square \)

*Fig(5)*: The region \( R_4 \) where \( b = -0.5 \)

**Lemma (4-12)**

There are no periodic points in \( R_5 \).
Proof: Let $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathbb{R}_5$, so $y_0 < -\frac{x_0 - x_0^2}{b}$, $-y_0^2 - by_0 < x_0 < -y_0^2$, $x_0 > y_0$, $by_0 + x_0^2 > 1$, we have $x_1 = -by_0 - x_0^2$, $y_1 = x_0$. We claim that $\mathbb{R}_5$ maps into $\mathbb{R}_4$, since $x_0 \geq y_0$, we have $-x_0^2 - bx_0 \geq -x_0^2 - by_0$ also since $-x_0^2 - by_0 = x_1$, $x_0 = y_1$, we have $x_1 \leq -y_1^2 - by_1$. \hfill (4.20)

On the other hand, since we have $x_0 < -y_0^2$, then $\sqrt{-x_0} > \sqrt{y_0^2} = |y_0| = -y_0$ so, $by_0 < -b\sqrt{-x_0}$, hence $by_0 + x_0^2 < x_0^2 - b\sqrt{-x_0}$, thus $-x_1 < y_1^2 - b\sqrt{-y_1}$. \hfill (4.21)

To show that $y_1 < \frac{-x_1 - x_1^2}{b}$, since $by_0 + x_0^2 > 1$, we have $x_0 < -1$, so $x_1 > -x_1$. Since $bx_0 > 0$, we get $bx_0 > -x_1 - x_1^2$, that is $x_0 < \frac{-x_1 - x_1^2}{b}$ but $y_1 = x_0$, so $y_1 < \frac{-x_1 - x_1^2}{b}$. \hfill (4.22)

Now from (4.20), (4.21) and (4.22), our claim is true, so by lemma (4-11) there are no periodic points in $\mathbb{R}_5$. □

![Fig(6): The region $\mathbb{R}_5$ where $b = -0.5$](image)

**Lemma (4-13)**

The region $\mathbb{R}_6$ maps into $\mathbb{R}_2 \cup \mathbb{R}_7$ under backward iteration.

Proof: Let $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ be any element in $\mathbb{R}_6$. We must show that $H^{-1} \left( \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right) \in \mathbb{R}_2 \cup \mathbb{R}_7$.

From (4.3) $x_{-1} = y_0, y_{-1} = \frac{-x_0 - y_0^2}{b}$, since $y_0 < -1 - b$, we have $x_{-1} < -1 - b$. \hfill (4.23)
Since \( x_0 > -y_0^2 - by_0 \), we have \( y_0 < \frac{-x_0 - y_0^2}{b} \), that is \( x_{-1} < y_{-1} \). 

(4.24)

Also, since \( x_0 > b\sqrt{-y_0 - y_0^2} \), we have \( \frac{x_0 + y_0^2}{b} < \sqrt{-y_0} \), so \( -y_{-1} < \sqrt{-y_0} \) thus, \( y_0 < -y_{-1}^2 \), hence \( x_{-1} < -y_{-1}^2 \). 

(4.25)

From \( x_0 \geq y_0 \), we have \( \frac{-x_0 - y_0^2}{b} \geq \frac{-y_0 - y_0^2}{b} \), hence \( y_{-1} \geq \frac{-x_{-1} - x_{-1}^2}{b} \). 

(4.26)

Now from (3.38), (3.39), (3.40) and (3.41) \( \left( \begin{array}{c} x_{-1} \\ y_{-1} \end{array} \right) \in R_2 \cup R_7 \), that is \( H^{-1} \left( \begin{array}{c} x_0 \\ y_0 \end{array} \right) \) in \( R_2 \cup R_7 \).  

□

![Image](image.png)

**Fig.(7): The region \( R_6 \) where \( b = -0.5 \)

**Lemma (4-14)**

There are no periodic points in \( R_7 \).

Proof: Let \( \left( \begin{array}{c} x_0 \\ y_0 \end{array} \right) \) be any element in \( R_7 \), since \( y_0 \geq \frac{-x_0 - x_0^2}{b} \), the graph of the map \( y = \frac{-x-x^2}{b} \) intersects x-axis at point (-1,0), thus \( x_0 > -1 \), so \( b \frac{-x_0 - y_0^2}{b} + y_0^2 < 1 \), that is \( by_{-1} + y_{-1}^2 < 1 \). 

(4.27)

Since \( -x_0 > 1 + b \), \( y_0 > -1 - b \), we have \( -y_0^2 - x_0 > (1 + b) - (1 + b)^2 = b(-1 - b) \), hence \( \frac{-x_0 - y_0^2}{b} < -1 - b \), \( y_{-1} < -1 - b \). 

(4.28)
On the other hand, $x_0 < -y_0^2 - by_0$, thus $y_0 > \frac{-x_0 - y_0^2}{b}$, so $x_{-1} > y_{-1}$.

(4.29)

Also, since $x_0 > b\sqrt{-y_0 - y_0^2}$, we get $\frac{x_0 + y_0^2}{b} < \sqrt{-y_0}$, thus $-y_{-1} < \sqrt{-y_0}$ so $y_{-1}^2 < -y_0$, that is $x_{-1} < -y_{-1}^2$.

(4.30)

Hence, from (4.27), (4.28), (4.29) and (4.30) $R_7$ maps into $R_6$, by lemma (4-13) $R_6$ maps into $R_2 \cup R_7$, so there is no point $\left(\frac{x_0}{y_0}\right)$ in $R_7$ such that $H^{-2n-l}\left(\frac{x_0}{y_0}\right) \in R_7$. That is, there is no point $\left(\frac{x_0}{y_0}\right)$ in $R_7$ such that $H^{-2n-l}\left(\frac{x_0}{y_0}\right) = \left(\frac{x_0}{y_0}\right)$ hence there are no periodic points of odd period in $R_7$. To show that there are no periodic points of even period in $R_7$, let $P=\{\left(\frac{x_0}{y_0}\right) \in R_7 : H^{-2n-l}\left(\frac{x_0}{y_0}\right) \in R_6, H^{-2n}\left(\frac{x_0}{y_0}\right) \in R_7 \forall n \in N, \text{if } \left(\frac{x_0}{y_0}\right) \in P \}$, we claim that the sequence $\langle y_{-2n} \rangle$ is strictly increasing, since $y_0 > -1 - b$, we have $y_0(-1 - b) < (1 + b)^2$. (4.31)

Also $y_0^2 < (1 + b)^2$, $-x_0 > 1 + b$, so $-x_0 - y_0^2 > -b(1 + b)$, hence $\frac{-x_0 - y_0^2}{b} < -1 - b$ that is

$\left(\frac{-x_0 - y_0^2}{b}\right) > (1 + b)^2$. (4.32)

Now from (4.31), (4.32), we get that $\frac{-x_0 - y_0^2}{b} > y_0(-1 - b) = -y_0 - by_0$, so $by_0 > -y_0 - y_{-1}^2$, that is

$y_0 < \frac{-x_{-1} - y_{-1}^2}{b} = y_{-2}$. In general, since $\left(\frac{x_{-2n}}{y_{-2n}}\right) \in R_7$, we have, $y_{-2n} > -1 - b$, $x_{-2n} < -1 - b$, so

$y_{-2n}(-1 - b) < (1 + b)^2$, $y_{-2n}^2 < (1 + b)^2$. Hence $-x_{-2n} - y_{-2n}^2 > -b(1 + b)$, that is

$\left(\frac{-x_{-2n} - y_{-2n}^2}{b}\right) > (1 + b)^2 > (-1 - b)y_{-2n}$. So $y_{-2n}^2 > (-1 - b)y_{-2n} = -y_{-2n} - by_{-2n} = -x_{-2n-1} - by_{-2n}$, that is, we get that $by_{-2n} > -x_{-2n-1} - y_{-2n-1}^2$, hence $y_{-2n} < \frac{-x_{-2n-1} - y_{-2n-1}^2}{b} = y_{-2n-2}$, for all $n \geq 1$.

So, $\langle y_{-2n} \rangle$ is strictly increasing sequence in $R_7$, so there are no periodic points of even period in $P$. □
Lemma (4-15)

There are no periodic points in $R_\alpha$.

Proof: by lemma (4-13), $R_\alpha$ maps into $R_2 \cup R_\gamma$, there are no periodic points in $R_2 \cup R_\gamma$, so there are no periodic points in $R_\alpha$. □

Theorem (4-16)

There are no periodic points of $H_{0,b}$ in $\mathbb{R}^2$ other than fixed points.

Proof: If $x_0 > -1 - b, y_0 = -1 - b$, we have $-x_0^2 > -(1 + b)^2$, $-by_0 = b + b^2$, hence $-x_0^2 - by_0 = b + b^2 - 1 - b^2 - 2b = -1 - b$, so $x_1 > -1 - b, y_1 = x_0 > -1 - b$, all points $(x, y)$ in $R_3$ satisfy the equation $y < \frac{-x^2}{b}$, if $y_0 > -1 - b, x_0 = -1 - b$, we get that $-by_0 = b + b^2$, $x^2 = (1 + b)^2$ so $-x^2 - by > -1 - b$, that is $x_1 > -1 - b, y_1 = x_0 = -1 - b$

Hence from theorem (3-3), and previous lemmas (4-1),(4-2),..., (4-14), (4-15), since a periodic point under forward iteration implies one under backwards iteration, and vice versa. □

References


[4] Hayes, S. and Wolf, C. Dynamics of a One-Parameter Family of Henon Maps. Department of Mathematics, University of Wichita, KS 67260, E-mail address: c.wolf@math.wichita.edu.
