The dynamics of Henon Map $H_{a,b}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}a-by-x^2\\x\end{pmatrix}$ Where $a=0, -1<b<0$

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Abstract

We prove the Henon map $H_{0,b}$ $,-1<b<0$, has no periodic point for any period in the plane,.$H_{0,b}$ has attracting fixed point $\begin{pmatrix}0\\0\end{pmatrix}$ and a saddle fixed point..There exists an open set about $\begin{pmatrix}0\\0\end{pmatrix}$ in which all points tend to $\begin{pmatrix}0\\0\end{pmatrix}$ under forward iteration of $H_{0,b}$.

1-Introduction

About 30 years ago the French astronomer –mathematician Michel-Henon was searched for a simple two-dimensional map possessing special properties of more complicated systems .The result was a family of maps denoted by $H_{a,b}$ given by $H_{a,b}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1-ax^2+y\\bx\end{pmatrix}$ where a,b are real numbers .These maps defined in above are called Henon maps [3]. Henon map $H_{a,b}:R^2 \longrightarrow R^2$.The importance of Henon's map for the dynamics of all polynomial diffeomorphisms was recognized by S.Friedland and J.Milnor , who proved that every quadratic polynomial diffeomorphism is conjugate to a Henon map or to an elementary maps have trivial dynamics .Henon map is normal form for all quadratic polynomial diffeomorphism with non- trivial dynamic.[4].This work we, care this form of Henon map $H_{a,b}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}a-by-x^2\\x\end{pmatrix}$.

To determine the Henon map $H_{0,b}$ $,-1<b<0$, has no periodic point for any period in the plane by dividing the plane to some region which are shown in section four we conclude that there exists an open set about $\begin{pmatrix}0\\0\end{pmatrix}$ in which all points tend to $\begin{pmatrix}0\\0\end{pmatrix}$ under forward iteration of H .We found these regions $Q_1, Q_{2,1}, Q_{2,2}, Q_3, Q_4$ and $R_1, R_2, R_3, R_4, R_5, R_6, R_7$, we prove there are no periodic point for Henon map in the plane .Also, we prove that the norm of iteration of Henon map $\mathrm{H}_{0,b}$ tends to infinity for some regions shown in proposition (4-2) and (4-4).

In section two, we recall some necessary definitions and theorems. In section three,we introduce forward and backward iteration of Henon map $\mathrm{H}_{a,b}$, where $a = 0$,$-1<b<0$ .In section four ,we study the type of fixed point of Henon map $\mathrm{H}_{0,b}$ with a basin of attraction of one of fixed points .In section five we study non existence of periodic point of Henon map in R^2 by finding some region in R^2 .At the last ,we introduce forward and backward iteration of Henon map $\mathrm{H}_{a,b}$, where $a = 0, b > 0$. we conclude that there exists an open set about $\begin{pmatrix}0\\0\end{pmatrix}$ in which all points tend to $\begin{pmatrix}0\\0\end{pmatrix}$ under forward iteration of H ., we prove that the norm of iteration of Henon map $\mathrm{H}_{0,b}$ tends to infinity for some regions

2-Preliminaries

The purpose of this section is to introduce some definitions and theorems necessary for this research, we recall some fundamental definitions and necessary theorems. Let **A** be an $n \times n$ matrix **.**The real number λ is called **eigen value** of **A** if there exists a non zero vector X in R^n such that **AX=**λ **X** (2.1)

Every non zero vector X satisfying (2.1) is called **an eigen vector** of A **associated with the eigen value** λ .In our work, we write the mapF: $\mathrm{V} \longrightarrow \mathrm{R}^2$ where $\mathrm{V} \subseteq \mathrm{R}^2$, such that $\mathrm{F}\begin{pmatrix}x\\y\end{pmatrix} = \begin{bmatrix} f\begin{pmatrix}x\\y\end{pmatrix} \\ g\begin{pmatrix}x\\y\end{pmatrix}\end{bmatrix}$; $\begin{pmatrix}x\\y\end{pmatrix} \in \mathrm{V}$.Also,the forward orbit of a vector $\begin{pmatrix}x_0\\y_0\end{pmatrix}$ is the set of points $\begin{pmatrix}x_0\\y_0\end{pmatrix}$,$\mathrm{F}\begin{pmatrix}x_0\\y_0\end{pmatrix}$,$\mathrm{F}^2\begin{pmatrix}x_0\\y_0\end{pmatrix}$ $\mathrm{F}^3\begin{pmatrix}x_0\\y_0\end{pmatrix}$,…and denoted by $\mathrm{O}^+\begin{pmatrix}x_0\\y_0\end{pmatrix}$.If F a homeomorphism ,we may define the full orbit of $\begin{pmatrix}x_0\\y_0\end{pmatrix}$,$\mathrm{O}\begin{pmatrix}x_0\\y_0\end{pmatrix}$,as the set of points $\mathrm{F}^n\begin{pmatrix}x_0\\y_0\end{pmatrix}$ for $n \in Z$,and the backward orbit of $\begin{pmatrix}x_0\\y_0\end{pmatrix}$, $\mathrm{O}^-\begin{pmatrix}x_0\\y_0\end{pmatrix}$, as the set of points $\mathrm{F}^{-1}\begin{pmatrix}x_0\\y_0\end{pmatrix}$,$\mathrm{F}^{-2}\begin{pmatrix}x_0\\y_0\end{pmatrix}$, $\mathrm{F}^{-3}\begin{pmatrix}x_0\\y_0\end{pmatrix}$,…,where we have $\mathrm{F}^n\begin{pmatrix}x_0\\y_0\end{pmatrix} = \mathrm{F}^{n-1}\begin{pmatrix}x_1\\y_1\end{pmatrix} = \begin{pmatrix}x_n\\y_n\end{pmatrix}$.

Definition (2-1)[6] Any pair $\begin{pmatrix} p \\ q \end{pmatrix}$ for which $f\begin{pmatrix} p \\ q \end{pmatrix}=p$, $g\begin{pmatrix} p \\ q \end{pmatrix}=q$ (2.2)

is called **a fixed point** of the two dimensional dynamical system

Definition (2-2)[1] Let F: $R^n \longrightarrow R^n$ be a map, $x_0 \in R^n$. The point x_0 is a **periodic point** of **period** m if $F^m(x_0)=x_0$. The least positive integer m for which $F^m(x_0)=x_0$ is called the prime period of x_0 .

Example (2-3) Let F: $R^2 \longrightarrow R^2$ is given by $F\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} y \\ x \end{pmatrix}$, then $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is a periodic point of period 2 of F

Definition (2-4)[2] Let $\begin{pmatrix} p \\ q \end{pmatrix}$ be a fixed point of F, then $\begin{pmatrix} p \\ q \end{pmatrix}$ is **attracting** fixed point if and only if there is a disk centered at $\begin{pmatrix} p \\ q \end{pmatrix}$ such that $F^{(n)}\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} p \\ q \end{pmatrix}$ for every $\begin{pmatrix} x \\ y \end{pmatrix}$ in the disk as $n \longrightarrow \infty$. By contrast $\begin{pmatrix} p \\ q \end{pmatrix}$ is **repelling** fixed point if and only if there is a disk centered at $\begin{pmatrix} p \\ q \end{pmatrix}$ such that $\left\| F\begin{pmatrix} u \\ v \end{pmatrix} - F\begin{pmatrix} p \\ q \end{pmatrix} \right\| > \left\| \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} p \\ q \end{pmatrix} \right\|$ for every $\begin{pmatrix} u \\ v \end{pmatrix}$ in the disk for which $\begin{pmatrix} u \\ v \end{pmatrix} \neq \begin{pmatrix} p \\ q \end{pmatrix}$.

Theorem (2-5)[2] Let $\begin{pmatrix} p \\ q \end{pmatrix}$ be a fixed point of F. Assume that $DF\begin{pmatrix} p \\ q \end{pmatrix}$ exists, with eigen values λ_1 , λ_2, then :

(1) $\begin{pmatrix} p \\ q \end{pmatrix}$ is an attracting fixed point, if λ_1 and λ_2 are less than one in absolute value .

(2) $\begin{pmatrix} p \\ q \end{pmatrix}$ is repelling fixed point, if λ_1 and λ_2 are greater than one in absolute value .

(3) $\begin{pmatrix} p \\ q \end{pmatrix}$ is saddle point, if one of λ_1 , λ_2 is larger and the other is less than one in absolute value.

Example (2-6)[2] Let $F\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} y \\ a\sin(x)-y \end{pmatrix}$ then $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is an attracting fixed point if $\frac{-1}{4}<a<0$, $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a repelling fixed point if $a>2$, $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is saddle fixed point if $0<a<2$.

Definition (2-7)[1]

Let $V,S \subseteq R^2$,F: V$\longrightarrow$V and G:S$\longrightarrow$S be two maps .Then F and G are said to be **topologically conjugate** if there exist a homeomorphism H:V$\longrightarrow$S such that $H \circ F = G \circ H$ .The homeomorphism H is called **a topological conjugacy**

Example (2-8)[6]

Let $Q_4(x)=4x(1-x)$ for $0\le x\le 1$ and $T(x)=\begin{cases} 2x & for \quad 0\le x\le \frac{1}{2} \\ 2(1-x) & for \quad \frac{1}{2}< x\le 1 \end{cases}$.

Q_4 is topologically conjugate to T(x). Where topological conjugacy is $\sin^2\frac{\pi}{2}x$.

Definition (2-9)[3]

Let $\begin{pmatrix} p \\ q \end{pmatrix}$ be a fixed point of F .The **basin of attraction** of $\begin{pmatrix} p \\ q \end{pmatrix}$ consists of all $\begin{pmatrix} x \\ y \end{pmatrix}$ such that $F^n\begin{pmatrix} x \\ y \end{pmatrix}\longrightarrow\begin{pmatrix} p \\ q \end{pmatrix}$ as $n\longrightarrow\infty$.

Theorem (2-10)[1]

Suppose that F has an attracting fixed point at $\begin{pmatrix} p \\ q \end{pmatrix}$.Then there is an open set about $\begin{pmatrix} p \\ q \end{pmatrix}$, in which all points tend to $\begin{pmatrix} p \\ q \end{pmatrix}$ under forward iteration of F.

Proposition (2.11)[1],[5]

Let H: $R^2 \longrightarrow R^2$ be a Henon map and b be any fixed real number. Then

(1) $JH_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}=b$, $\forall$ x, y in R.

(2) If $x^2-b\ge 0$, then the eigen values of $DH_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$ are the real numbers - $x\pm\sqrt{x^2-b}$.(3) If $b\ne 0$, $H_{a,b}$ is one-to-one map

(4) The Henon map $H_{a,b}$ is C^∞.

(5) If $b\ne 0$ then $H_{a,b}$ has an inverse.

(6) If $b\ne 0$ $H_{a,b}$ is diffeomorphism .

Proposition (2.12)[1]

For each value of b in R, there exists b_0 in $R^-U\{0\}$ such that, if $a>b_0$ the Henon map has two fixed points.

3-Type of fixed points with basin of attraction

Our goal of this section is to determine type of fixed points of $H_{0,b}$ with the basin of attraction of a fixed point $\begin{pmatrix}0\\0\end{pmatrix}$. We will use maximum norm, where the maximum norm of $\begin{pmatrix}x\\y\end{pmatrix}$ = $\left\|\begin{pmatrix}x\\y\end{pmatrix}\right\|$=max$\{|x|, |y|\}$.

Proposition (3-1)

If $-1<b<0$, then $H_{0,b}$ has attracting fixed point $\begin{pmatrix}0\\0\end{pmatrix}$ and a saddle fixed point $\begin{pmatrix}-(b+1)\\-(b+1)\end{pmatrix}$.

Proof: Since $a=0$ and $-1<b<0$. By proposition (2.12) $P_+ = \begin{pmatrix}0\\0\end{pmatrix}$, $P_- = \begin{pmatrix}-(b+1)\\-(b+1)\end{pmatrix}$ are fixed points for $H_{0,b}$.By proposition (2.11) $\lambda_{1,2}=\pm\sqrt{-b}$ are eigen values of $DH_{0,b}\begin{pmatrix}x\\y\end{pmatrix}$ at P_+ . Since $-1<b<0$ then $0<\sqrt{-b}<1$ and $|\lambda_1|=|\lambda_2|<1$, so $\begin{pmatrix}0\\0\end{pmatrix}$ is attracting fixed point. To show that P_- is a saddle fixed point . From proposition (2.11) , we have two eigen values, $\lambda_1=-(b+1)+\sqrt{(b+1)^2-b}$, $\lambda_2=-(b+1)-\sqrt{(b+1)^2-b}$ (3.1)

Since $b^2+b+1<(b+2)^2$ thus $\sqrt{b^2+b+1}<\sqrt{(b+2)^2}=b+2$ hence $-(b+1)+\sqrt{b^2+b+1}<1$ (3.2)

On the other hand, since $\sqrt{b^2+b+1}=\sqrt{(b+1)^2-b}>\sqrt{(b+1)^2}=|b+1|=b+1$

so $-(b+1)+\sqrt{b^2+b+1}>0$, hence by (3.2) $\left|-(b+1)+\sqrt{(b+1)^2-b}\right|=|\lambda_2|<1$. (3.3)

For λ_1 since $-1<b<0$ we have $\sqrt{b^2+b+1}>\sqrt{b^2}=|b|=-b$. (3.4)

By adding $(b+1)$ for both sides of (3.4), we get $(b+1)$ + $\sqrt{b^2+b+1}$ >1

then $\left|(b+1)+\sqrt{(b+1)^2-b}\right|=|\lambda_1|>1$. (3.5) Hence from (3.4), (3.5) and theorem (2-5), P_- is a saddle fixed point. □

Remark (3-2) In view of theorem (2-10), there exists an open set about $\begin{pmatrix}0\\0\end{pmatrix}$ in which all points tend to $\begin{pmatrix}0\\0\end{pmatrix}$ under forward iteration of $H_{0,b}$.The next theorem shows this open set.

Theorem(3-3) Suppose S= $\{\begin{pmatrix}x\\y\end{pmatrix}: x,y\in R, |x|\le 1-|b|, |y|\le 1-|b|, |b|<1\}$ then for all $\begin{pmatrix}x\\y\end{pmatrix}\in S^\circ$,$H^n_{0,b}\begin{pmatrix}x\\y\end{pmatrix}\longrightarrow\begin{pmatrix}0\\0\end{pmatrix}$ as $n\longrightarrow\infty$.

Proof: We claim that $H_{0,b}(S^\circ)\subset S^\circ$, to show this , let $\begin{pmatrix}x_1\\y_1\end{pmatrix}$ be any element in $H_{0,b}(S^\circ)$ then there is $\begin{pmatrix}x\\y\end{pmatrix}$ in S° such that $H_{0,b}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x_1\\y_1\end{pmatrix}=\begin{pmatrix}-by-x^2\\x\end{pmatrix}$ .From this $|y_1|=|x|<1-|b|$ (3.6)

and $|x_1|=|by+x^2|\le|b||y|+|x|^2<|b|\ (1-|b|\ )+(1-|b|)^2=|b|-|b|^2+1-2|b|+|b|^2=1-|b|$ (3.7)

Now from (3.6), (3.7)$\begin{pmatrix}x_1\\y_1\end{pmatrix}\in S^\circ$. Let $r=\left\|\begin{pmatrix}x\\y\end{pmatrix}\right\|$, so $|x|\le r$, $|y|\le r$ and since$|x|\le 1-|b|$, $|y|\le 1-|b|$, then $r<1-|b|$. We denote $H_{0,b}{}^n\begin{pmatrix}x\\y\end{pmatrix}$ for $\begin{pmatrix}x\\y\end{pmatrix}$ in S°,by $\begin{pmatrix}x_n\\y_n\end{pmatrix}$,then$|y_2|=|x_1|=|by+x^2|\le|b||y|+|x|^2\le|b|\ r+r^2=r(|b|+r)$ (3.8)

$|x_2|=|by_1+x_1{}^2|\le|b|\ r+r^2(|b|+r)^2$ (3.9)

Now since $0<|b|+r<1$, we have $0<r^2\ (\ |b|+r\ )<r^2$, then $|b|\ r+r^2(|b|+r)^2<|b|\ r+r^2=r(|b|+r)$, hence by (3.8), (3.9) $|x_2|<r(|b|+r)$ (3.10)

Now we claim that $|x_{2n}|<r(|b|+r)^n$, $|y_{2n}|<r(|b|+r)^n$. (3.11)

We prove it by using mathematical induction from (3.8), (3.10),it is true for $n=1$, suppose it is true for k then $|x_{2k}|<r(|b|+r)^k$,$|y_{2k}|<r(|b|+r)^k$ to show that it is true for $k+1$,

$y_{2(k+1)}=-by_{2(k+1)-2}-x^2{}_{2(k+1)-2}\quad=-by_{2k}-x^2{}_{2k}$.

$|y_{2(k+1)}|=|by_{2k}+x_{2k}{}^2|\le|b||y_{2k}|+|x_{2k}|^2\le|b|\ r(|b|+r)^k+r^2(|b|+r)^{2k}$ (3.12)

Now $0<(\ |b|+r\ )^{\ k}<1$, so $r^2\ (\ |b|+r\ )^{\ k}+|b|\ r<|b|\ r+r^2=r(|b|+r)$

Hence $r^2\ (\ |b|+r\ )^{\ 2\ k}+|b|\ r\ (\ |b|+r\ )^{\ k}<r(|b|+r)^{\ k+1}$ (3.13)

Hence from (3.12),we get $|y_{2(k+1)}|<r(|b|+r)^{\ k+1}$ (3.14)

Also we have $x_{2(k+1)}=-by_{2(k+1)-1}\ -x^2_{2(k+1)-1}=-by_{2k+1}-x^2{}_{2k+1}=-bx_{2k}-y^2{}_{2(k+1)}$, since $x_n=y_{n+1}$

so $|x_{2(k+1)}|=|bx_{2k}+y^2{}_{2(k+1)}|<|b||x_{2k}|+|y_{2(k+1)}|^2<|b|\ r(|b|+r)^{\ k}+r^2(|b|+r)^{\ 2\ k}$ hence from (3.13) we get that $|x_{2(k+1)}|<r(|b|+r)^{\ k+1}$, hence it is true for $k+1$.Now $\left\|H_{0,b}{}^{2n}\begin{pmatrix}x\\y\end{pmatrix}\right\|=\left\|\begin{pmatrix}x_{2n}\\y_{2n}\end{pmatrix}\right\|$=max$\{|x_{2n}|,|y_{2n}|\}$,if $\left\|H_{0,b}{}^{2n}\begin{pmatrix}x\\y\end{pmatrix}\right\|$ $=|x_{2n}|$ then $\lim\limits_{n\longrightarrow\infty}\left\|H_{0,b}{}^{2n}\begin{pmatrix}x\\y\end{pmatrix}\right\|=\lim\limits_{n\longrightarrow\infty}|x_{2n}|=\lim\limits_{n\longrightarrow\infty}r(|b|+r)^{\ n}=0$, so by definition of maximum norm of $\left\|H_{0,b}{}^{2n}\begin{pmatrix}x\\y\end{pmatrix}\right\|$. $\lim\limits_{n\longrightarrow\infty}|y_{2n}|=0$ so $\mathrm{H}_{0,b}{}^{2n}\begin{pmatrix}x\\y\end{pmatrix}\longrightarrow\begin{pmatrix}0\\0\end{pmatrix}$ as $n\longrightarrow\infty$. (3.15)

If $\left\|H_{0,b}{}^{2n}\begin{pmatrix}x\\y\end{pmatrix}\right\|=|y_{2n}|$, as the same as (3.13), $\mathrm{H}_{0,b}{}^{2n}\begin{pmatrix}x\\y\end{pmatrix}\longrightarrow\begin{pmatrix}0\\0\end{pmatrix}$ as $n\longrightarrow\infty$ and since $\mathrm{H}_{0,b}(\mathrm{S}^\circ)\subset\mathrm{S}^\circ$ then $\forall\begin{pmatrix}x\\y\end{pmatrix}\in\mathrm{S}^\circ$, we have $\mathrm{H}_{0,b}\begin{pmatrix}x\\y\end{pmatrix}\in\mathrm{S}^\circ$ by (3.15), so $\mathrm{H}_{0,b}{}^{2n}\left(\mathrm{H}_{0,b}\begin{pmatrix}x\\y\end{pmatrix}\right)\longrightarrow\begin{pmatrix}0\\0\end{pmatrix}$ as $n\longrightarrow\infty$, so $\mathrm{H}_{0,b}{}^{2n+1}\begin{pmatrix}x\\y\end{pmatrix}\longrightarrow\begin{pmatrix}0\\0\end{pmatrix}$ as $n\longrightarrow\infty$, hence $\mathrm{H}_{0,b}{}^{n}\begin{pmatrix}x\\y\end{pmatrix}\longrightarrow\begin{pmatrix}0\\0\end{pmatrix}$ as $n\longrightarrow\infty$, $\forall\begin{pmatrix}x\\y\end{pmatrix}\in\mathrm{S}^\circ$. □

4- The Periodic Points for Henon Map Where $a=0,-1<b<0$

In this section, we will show that Henon map $\mathrm{H}_{0,b},-1<b<0$ has no periodic points other than fixed points in the plane. To prove this, we divide the proof to fifteen lemmas, thus we find some regions such that the union of all regions covers the plane. We prove that there is no periodic point in each of them until we get the main purpose. The regions are the following $Q_1,Q_{2,1},Q_{2,2},Q_3,Q_4$ and in Q_3 define the following regions $R_1,R_2,R_3,R_4,R_5,R_6,R_7$.

$Q_1=\{\begin{pmatrix}x\\y\end{pmatrix}:x\geq0,y\geq0\}$.

$Q_{2,1}=\{\begin{pmatrix}x\\y\end{pmatrix}:x<0,y\geq0,x\ \geq-y^{\ 2}\}$.

$Q_{2,2}=\{\begin{pmatrix}x\\y\end{pmatrix}:x<0,y>0,x\ <-y^{\ 2}\}$.

$$Q_4 = \{\begin{pmatrix} x \\ y \end{pmatrix}: \ x \geq 0, y < 0\}.$$

$$R_1 = \{\begin{pmatrix} x \\ y \end{pmatrix}: x \ \geq -y^2\}.$$

$$R_2 = \{\begin{pmatrix} x \\ y \end{pmatrix}: x \ \leq b\sqrt{-y} - y^2\}$$

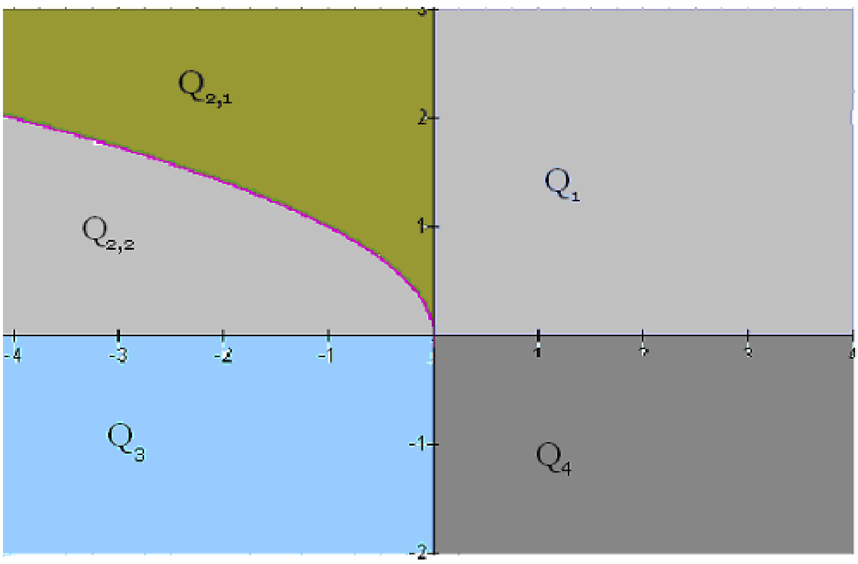
$$R_3 = \{\begin{pmatrix} x \\ y \end{pmatrix}: y < \frac{-x - x^2}{b}, -y^2 - by < x \ < -y^2, x < y\}.$$

$$R_4 = \{\begin{pmatrix} x \\ y \end{pmatrix}: y < \frac{-x - x^2}{b}, b\sqrt{-y} - y^2 < x \ \leq -by - y^2\}$$

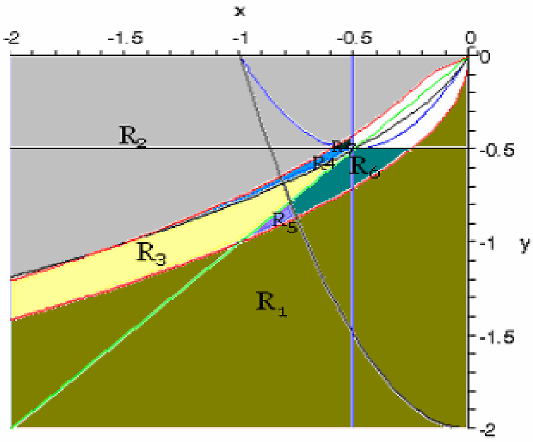
$$R_5 = \{\begin{pmatrix} x \\ y \end{pmatrix}: y < \frac{-x - x^2}{b}, -y^2 - by < x \ < -y^2, x \geq y, by + x^2 > 1\}.$$

$$R_6 = \{\begin{pmatrix} x \\ y \end{pmatrix}: y < -1 - b, x \ < -y^2, x \geq y, by + x^2 \leq 1, x > -y^2 - by, x > b\sqrt{-y} - y^2\}.$$

$$R_7 = \{\begin{pmatrix} x \\ y \end{pmatrix}: x < -1 - b, y > -1 - b, y > x, y \geq \frac{-x - x^2}{b} \ , -y^2 - by > x > b\sqrt{-y} - y^2\}.$$



Fig(1):Region Q_1 $, Q_{2,1}, Q_{2,2}, Q_3$ $, Q_4$



Fig(2):Region $R_1, R_2, R_3, R_4, R_5, R_6, R_7$

Lemma (4-1)

$H_{0,b}$ is topologically conjugate to $H^{-1}_{0.b^{-1}}$

Proof: Let G: $R^2 \longrightarrow R^2$ be a map defined by $G\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} by \\ bx \end{pmatrix}$. Clearly G is continuous and bijective , G^{-1} is continuous, so G is homeomorphism and $G \circ H^{-1}_{0.b^{-1}}\begin{pmatrix} x \\ y \end{pmatrix} = G\begin{pmatrix} y \\ \dfrac{-x-y^2}{b^{-1}} \end{pmatrix} = \begin{pmatrix} b\dfrac{-x-y^2}{b^{-1}} \\ by \end{pmatrix} = \begin{pmatrix} \dfrac{-x-y^2}{b^{-2}} \\ by \end{pmatrix}$

(4.1)

$$H_{0,b} \circ G\begin{pmatrix} x \\ y \end{pmatrix} = H_{0,b}\begin{pmatrix} by \\ bx \end{pmatrix} = \begin{pmatrix} -b(bx)-(by)^2 \\ by \end{pmatrix} = \begin{pmatrix} b^2(-x-y^2) \\ by \end{pmatrix} = \begin{pmatrix} \dfrac{-x-y^2}{b^{-2}} \\ by \end{pmatrix} . \quad (4.2)$$

From (4.1) and (4.2), $G \circ H^{-1}_{0.b^{-1}} = H_{0,b} \circ G$. Hence by definition **(**2-7**),** $H_{0,b}$ is topologically conjugate to $H^{-1}_{0.b^{-1}}$. □

Lemma (4-2)

There are no periodic points in Q_1 other than fixed points.

Proof: Let $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_1$, we define norm $\left\| \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right\|$ by $\left\| \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right\| = \sqrt{x_0^2 + y_0^2}$ From now for simply we refer to $H_{0,b}$ as H and $H^{-1}_{0.b^{-1}}$ as H^{-1} $H^{-n}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_{-(n+1)} \\ y_{-(n+1)} \end{pmatrix} = \begin{pmatrix} y_{-n} \\ \frac{-1}{b}(x_{-n} + y_{-n}^2) \end{pmatrix}$ (4.3)

We have $-1 < b < 0$, $x_{-1} = y_0$, $y_{-1} = \frac{-1}{b}(x_0 + y_0)$ then $y_{-1} = \frac{-1}{b}(x_0 + y_0) > \frac{-1}{b} x_0 > x_0$

$y_{-1} > x_0$ and since y_{-1}, x_0 are non negative $y_{-1}^2 > x_0^2$.That is $y_{-1}^2 + y_0^2 > x_0^2 + y_0^2$

thus $y_{-1}^2 + x_{-1}^2 > x_0^2 + y_0^2$ so $\sqrt{y_{-1}^2 + x_{-1}^2} > \sqrt{x_0^2 + y_0^2}$,

hence $\left\| \begin{pmatrix} x_{-1} \\ y_{-1} \end{pmatrix} \right\| > \left\| \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right\|$, that is $\left\| H^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right\| > \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$. (4.4)

Since $x_{-1} = y_0 \geq 0$, $y_{-1} = \frac{-1}{b}(x_0 + y_0) \geq 0$,we have $H^{-1}(Q_1) \subset Q_1$, then (4.4) is true for all $n \in Z^+$,hence

$\left\| H^{-n}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right\| > \left\| H^{-n+1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right\| \quad \forall n \in Z^+$, (4.5)

so the norms of the points in the orbit are strictly increasing. Hence there are no periodic point in the region Q_1. □

Lemma (4-3)

There are no periodic points in $Q_{2,1}$.

Proof: Let $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_{2,1}$, since $H^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} y_0 \\ \frac{-1}{b}(x_0 + y_0^2) \end{pmatrix}$, $y_0 > 0$ and $x_0 \geq -y_0^2$, hence $x_{-1} > 0, \frac{-1}{b}(x_0 + y_0^2) \geq 0$, so $x_{-1} > 0$, $y_{-1} \geq 0$, $\begin{pmatrix} x_{-1} \\ y_{-1} \end{pmatrix} \in Q_1$, that means $Q_{2,1}$ maps into Q_1 then by lemma (4-.2) there are no periodic point for H in $Q_{2,1}$. □

Lemma (4-4)

There are no periodic points of even period in $Q_{2,2}$.

Proof: Let $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_{2,2}$.From (4.3),since $y_{-n} = \frac{-1}{b}(x_{-(n-1)} + y_{-(n-1)}^2)$, $x_{-n} = y_{-(n-1)}$. Since $x_0 < 0, y_0 \geq 0$,so $x_{-1} = y_0 > 0$, $y_{-1} = \frac{-1}{b}(x_0 + y_0^2) < 0$, hence $x_{-2} = y_{-1} < 0$, $y_{-2} = \frac{-1}{b}(x_{-1} + y_{-1}^2) > \frac{-1}{b} x_{-1}$. Since $-1 < b < 0$,

$\frac{-1}{b}x_{-1} > x_{-1} = y_0 > 0$,hence $y_{-2} > y_0 > 0$ .In general $y_{-n} > \frac{-1}{b}x_{-(n-1)} > x_{-(n-1)} = y_{-(n-2)}$, so $\langle y_{-2n}\rangle$ is strictly increasing sequence ,$H^{-2n}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ in $Q_{2,1} \cup Q_{2,2}$ then we have two cases.

Case 1: If $H^{-2n}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_{2,1}$, then by lemma (4-3) there are no periodic points of even period for H.

Case 2: If $H^{-2n}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_{2,2}$ and since $\langle y_{-2n}\rangle$ is strictly increasing sequence so there are no periodic points of even period for H. □

Lemma (4-5)

There are no periodic points of odd period in Q_4.

Proof: Let $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_4$ since $x_{-1} = y_0 < 0$, $y_{-1} = \frac{-1}{b}(x_0 + y_0^{\,2}), x_0 > 0$ so $y_{-1} = \frac{-1}{b}(x_0 + y_0^{\,2}) > \frac{-1}{b}x_0 > x_0$, hence $y_{-1} > 0$,from (4.3) $x_{-(n+1)} = y_{-n}$, we have $x_{-2} > 0$,then $y_{-3} = \frac{-1}{b}(x_{-2} + y_{-2}^{\,2}) > \frac{-1}{b}x_{-2} > x_{-2} = y_{-1} > 0$,hence $x_{-4} > 0$ by the same way

$$y_{-5} = \frac{-1}{b}(x_{-4} + y_{-4}^{\,2}) > \frac{-1}{b}x_{-4} > x_{-4} = y_{-3} > 0, \text{ hence } y_{-(2n+1)} > y_{-(2n-1)} > \ldots > y_{-5} > y_{-3} > y_{-1} > 0 \qquad (4.6)$$

Either $x_{-2n-1} > 0$ or $x_{-2n-1} < 0$ so $H^{-2n-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_1$ or $H^{-2n-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_2$

Case 1: If $H^{-2n-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_1$, by lemma (4-2), there are no periodic point of odd period

Case 2: If $H^{-2n-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_2$ since $\langle y_{-2n-1}\rangle$ is strictly increasing sequence there are no periodic points of odd period . □

Lemma (4-6)

There are no periodic points of even period in Q_4.

Proof: Suppose there exists $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ in Q_4 such that $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ is a periodic point of even period. That is there is $2n$ in Z^+, $H^{-2n}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$. (4.7)

Since $x_{-n} = y_{-n-1}, y_{-n} = \frac{-1}{b}(x_{-(n-1)} + y^2_{-(n-1)})$,then either $H^{-2n-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_{2,1}$ or $H^{-2n-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_1$ for all $n \in N$ .If $H^{-2n-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_{2,1}$,from (4.7) $H^{-2n}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ so $H^{-2n-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = H^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, hence $H^{-2n}(H^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}) = H^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, $H^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_{2,1}$ so $Q_{2,1}$ has a periodic point which is contradiction .If $H^{-2n-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_{2,2}$, in the same way, $H^{-2n}(H^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}) = H^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, $H^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_1$,that means there is a periodic point of even period in Q_1 which is contradiction ,so there are no periodic points of even period in Q_4 . □

Lemma (4-7)

There are no periodic points of odd period in $Q_{2,2}$.

Proof**:** Suppose that there is a periodic point $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ of odd period m .That is there exists at least positive integer n such that $m = 2n+1$, and $H^{-2n-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, since $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_{2,2}$, we have $x_0 < -{y_0}^2, x_{-1} = y_0 \geq 0$, also $y_{-1} = \frac{-1}{b}(x_0 + {y_0}^2) < 0$,hence $H^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_4$, that is $H^{-1}(H^{-2n-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}) = H^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$,also $H^{-2n-1}(H^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}) = H^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ hence $\begin{pmatrix} x_{-1} \\ y_{-1} \end{pmatrix}$ is a periodic point of odd period in Q_4 which is contradiction by lemma (4-5),hence there are no periodic points of odd period in $Q_{2,2}$.

Lemma (4-8)

There are no periodic points in R_1.

Proof: Let $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ be any element in R_1, then $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_3, x_0 \geq -{y_0}^2$,that is $y_0 < 0$

$x_0 < 0, x_0 \geq -{y_0}^2$, so $x_{-1} = y_0 < 0, y_{-1} = \frac{-1}{b}(x_0 + {y_0}^2) \geq 0$, hence $\begin{pmatrix} x_{-1} \\ y_{-1} \end{pmatrix} \in Q_2$ .That is $H^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_2$ that means R_1 maps into Q_2, so by lemma (4-4) and lemma (4-5) there are no periodic points in R_1. □

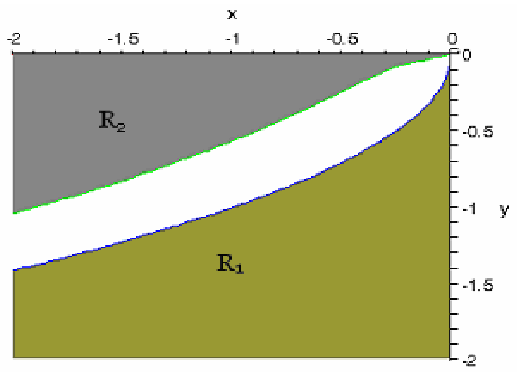
Lemma (4-9)

There are no periodic points in R_2.

Proof : Let $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in R_2$ then $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_3, x_0 \le b\sqrt{-y_0} - y_0^2$,that is $y_0 < 0, x_0 < 0$

$x_0 \le b\sqrt{-y_0} - y_0^2$ .Thus $x_0 + y_0^2 \le b\sqrt{-y_0}$ so $\frac{-1}{b}(x_0 + y_0^2) \le -\sqrt{-y_0}$, that is

$y_{-1} \le -\sqrt{-y_0}$ .Hence $y_{-1}^2 \ge -y_0$, that is $y_{-1}^2 \ge -x_1$ so $-y_{-1}^2 \le x_1$, hence $H^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in R_1$ .Hence R_2 maps into R_1, by lemma (4-8) ,there are no periodic point in R_2 . □



Fig(3):The region R1,R2 where b=-0.5

Lemma (4-10)

There are no periodic point in R_3.

Proof: Let $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in R_3$, so $y_0 < \frac{-x_0 - x_0^2}{b}, -y_0^2 - by_0 < x_0 < -y_0^2, x_0 < y_0$,we have $x_1 = -by_0 - x_0^2, y_1 = x_0$,

since $-x_0 < y_0^2 + by_0, x_0^2 > y_0^2$, we get that $y_0^2 + by_0 < x_0^2 + by_0$, so $-x_0 < x_0^2 + by_0$, that is $, x_0 > x_1$,since $x_0 = y_1$, we have $y_1 > x_1$. (4.8)

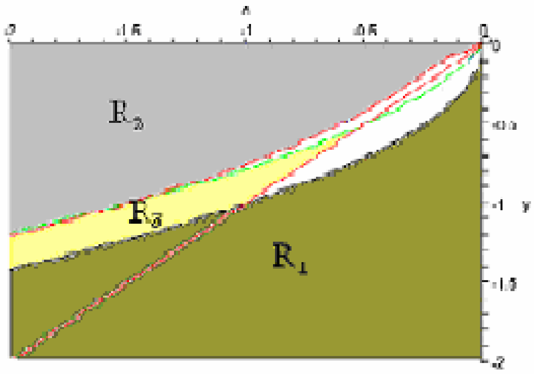
Suppose that $-x_1 \ge y_1^2 + by_1$, that is $, -x_1 \ge x_0^2 + bx_0, bx_0 > by_0$, so $-x_1 > x_0^2 + by_0 = -x_1$ which is contradiction, hence $x_1 > -y_1^2 - by_1$. (4.9)

Clearly, since $-by_0 < 0$, then $-by_0 - x_0^2 < -x_0^2$,hence $x_1 < -y_1^2$. (4.10)

To show that $y_1 < \dfrac{-x_1 - x_1^2}{b}$, since $x_0 < y_0, b < 0$, we have $-x_0^2 - by_0 > -x_0^2 - bx_0$, so $x_1 > -x_0^2 - bx_0 = -x_0^2 - by_1$. (4.11)

From $y_0 < \dfrac{-x_0 - x_0^2}{b}$, we have $by_0 > -x_0 - x_0^2$, hence $x_0 > x_1$, since $x_0 < 0$, $x_0^2 < x_1^2$, thus $-x_0^2 - by_1 > -x_1^2 - by_1$ then from (4.11), we get that $x_1 > -x_1^2 - by_1$, that is $by_1 > -x_1 - x_1^2$, hence $y_1 < \dfrac{-x_1 - x_1^2}{b}$. (4.12)

Now from (4.9),(4.10), (4.11)and (4.12), we can say that $H(R_3) \subset R_3$, so $H^2(R_3) \subset R_3$ and so on $H^n(R_3) \subset H^{n-1}(R_3) \subset R_3$, where n is positive integer hence, $x_n < y_n, y_n < \dfrac{-x_n - x_n^2}{b}$, that is $by_n > -x_n - x_n^2$, so $x_n > -x_n^2 - by_n = x_{n+1}$, hence $\langle x_n \rangle$ is strictly decreasing sequence in R_3, so there are no periodic points in R_3. □



Fig(4):The region R3 where b=-0.5

Lemma (4-11)

There are no periodic point in R_4.

Proof: Let $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ be any element in R_4, then $y_0 < \dfrac{-x_0 - x_0^2}{b}$ (4.13)

and $b\sqrt{-y_0} - y_0^2 < x_0 \le -by_0 - y_0^2$. (4.14)

By (3.28) $by_0 > -x_0 - x_0^2$ or $x_0 > -by_0 - x_0^2 = x_1$, from (3.18) $y_1 = x_0$, so $y_1 > x_1$ (4.15)

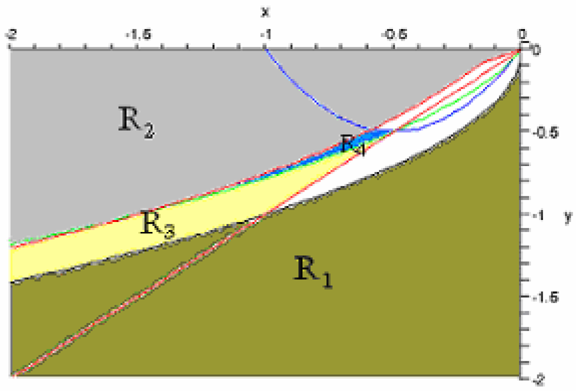
Since $x_1 = -by_0 - x_0^2, -by_0 < 0$, we have $x_1 < -y_1^2$. (4.16)

Now ,to show that $x_1 > -y_1^2 - by_1$, since $by_0 > -x_0 - x_0^2, -x_0 \geq by_0 + y_0^2$,we get $by_0 - x_0 > -x_0 - x_0^2 + y_0^2 + by_0$, so $-x_0^2 + y_0^2 < 0$, that is ,$\sqrt{x_0^2} > \sqrt{y_0^2}$ hence $|x_0| > |y_0|$,since $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in Q_3, |x_0| = -x_0, |y_0| = -y_0$,thus $x_0 < y_0$, so $y_1 < y_0$,since $b < 0$,we have $x_0^2 + by_0 < x_0^2 + by_1$, hence, $-x_1 < x_0^2 + by_1$. Now since $x_0 = y_1$, we get that $-x_1 < y_1^2 + by_1$ that is , $x_1 > -y_1^2 - by_1$. (4.17)

On the other hand , $x_0 < y_0$,thus ,$-x_0^2 - by_0 > -x_0^2 - bx_0$,hence we get that $x_1 > -x_0^2 - by_1$. (4.18)

From $by_0 > -x_0 - x_0^2$,then $x_0 > -x_0^2 - by_0$, so $x_0 > x_1$, x_0, x_1 are negative ,so $x_0^2 < x_1^2$, thus $-x_0^2 - by_1 > -x_1^2 - by_1$,hence from (4.18) $x_1 > -x_1^2 - by_1$, that is $y_1 < \dfrac{-x_1 - x_1^2}{b}$ (4.19)

Now from (4.15), (4.16), (4.17) and (4.19), we get that $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \in R_3$,so R_4 maps into R_3 ,then by lemma (4-10) there are no periodic points in R_4. □



Fig(5):The region R_4 where b=-0.5

Lemma (4-12)

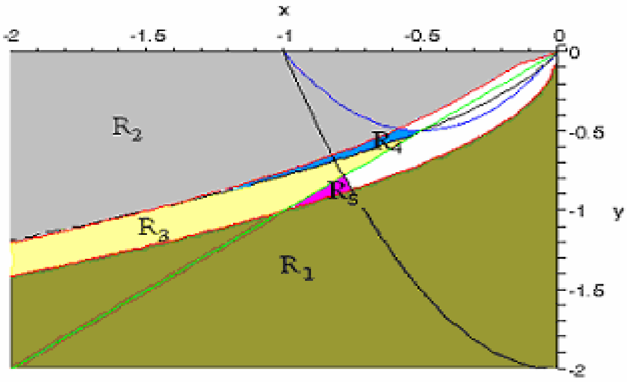
There are no periodic points in R_5.

Proof: Let $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in R_5$, so $y_0 < \frac{-x_0 - x_0^2}{b}$, $-y_0^2 - by_0 < x_0 < -y_0^2$, $x_0 > y_0$ $by_0 + x_0^2 > 1$, we have $x_1 = -by_0 - x_0^2$, $y_1 = x_0$. We claim that R_5 maps into R_4, since $x_0 \geq y_0$, we have $-x_0^2 - bx_0 \geq -x_0^2 - by_0$ also since $-x_0^2 - by_0 = x_1$ $x_0 = y_1$, we have $x_1 \leq -y_1^2 - by_1$. (4.20)

On the other hand, since we have $x_0 < -y_0^2$, then $\sqrt{-x_0} > \sqrt{y_0^2} = |y_0| = -y_0$ so, $by_0 < -b\sqrt{-x_0}$, hence $by_0 + x_0^2 < x_0^2 - b\sqrt{-x_0}$, thus $-x_1 < y_1^2 - b\sqrt{-y_1}$. (4.21)

To show that $y_1 < \frac{-x_1 - x_1^2}{b}$, since $by_0 + x_0^2 > 1$, we have $x_1 < -1$, so $x_1^2 > -x_1$. Since $bx_0 > 0$, we get $bx_0 > -x_1 - x_1^2$, that is $x_0 < \frac{-x_1 - x_1^2}{b}$ but $y_1 = x_0$, so $y_1 < \frac{-x_1 - x_1^2}{b}$. (4.22)

Now from (4.20), (4.21)and (4.22) ,our claim is true, so by lemma (4-11) there are no periodic points in R_5.□



Fig(6):The region R_5 where b=-0.5

Lemma (4-13)

The region R_6 maps into $R_2 \cup R_7$ under backward iteration .

Proof: Let $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ be any element in R_6. We must show that $H^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in R_2 \cup R_7$.

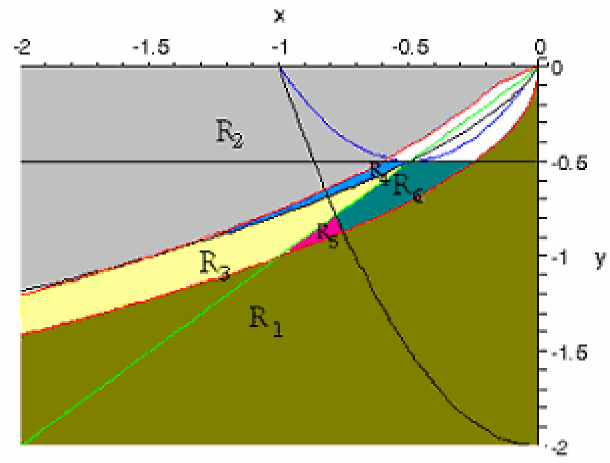
From (4.3) $x_{-1} = y_0, y_{-1} = \frac{-x_0 - y_0^2}{b}$, since $y_0 < -1 - b$, we have $x_{-1} < -1 - b$. (4.23)

Since $x_0 > -y_0^{\ 2} - by_0$, we have $y_0 < \dfrac{-x_0 - y_0^{\ 2}}{b}$,that is $x_{-1} < y_{-1}$. (4.24)

Also, since $x_0 > b\sqrt{-y_0} - y_0^{\ 2}$,we have $\dfrac{x_0 + y_0^{\ 2}}{b} < \sqrt{-y_0}$,so $-y_{-1} < \sqrt{-y_0}$ thus, $y_0 < -y_{-1}^{\ 2}$,hence $x_{-1} < -y_{-1}^{\ 2}$. (4.25)

From $x_0 \geq y_0$,we have $\dfrac{-x_0 - y_0^{\ 2}}{b} \geq \dfrac{-y_0 - y_0^{\ 2}}{b}$,hence $y_{-1} \geq \dfrac{-x_{-1} - x_{-1}^{\ 2}}{b}$. (4.26)

Now from (3.38), (3.39),(3.40)and (3.41) $\begin{pmatrix} x_{-1} \\ y_{-1} \end{pmatrix} \in R_2 \cup R_7$,that is $H^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ in $R_2 \cup R_7$. □



Fig(7):The region R_6 where b=-0.5

Lemma (4-14)

There are no periodic points in R_7.

Proof: Let $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$be any element in R_7, since $y_0 \geq \dfrac{-x_0 - x_0^{\ 2}}{b}$, the graph of the map $y = \dfrac{-x - x^2}{b}$ intersects x-axis at point (-1,0),thus $x_0 > -1$,so $b\dfrac{-x_0 - y_0^{\ 2}}{b} + y_0^{\ 2} < 1$, that is $by_{-1} + x_{-1}^{\ 2} < 1$. (4.27)

Since $-x_0 > 1 + b$, $y_0 > -1 - b$, we have $-y_0^{\ 2} - x_0 > (1+b) - (1+b)^2 = b(-1-b)$,hence $\dfrac{-x_0 - y_0^{\ 2}}{b} < -1 - b$, $y_{-1} < -1 - b$. (4.28)

On the other hand , $x_0 < -y_0^2 - by_0$, thus $y_0 > \dfrac{-x_0 - y_0^2}{b}$, so $x_{-1} > y_{-1}$. (4.29)

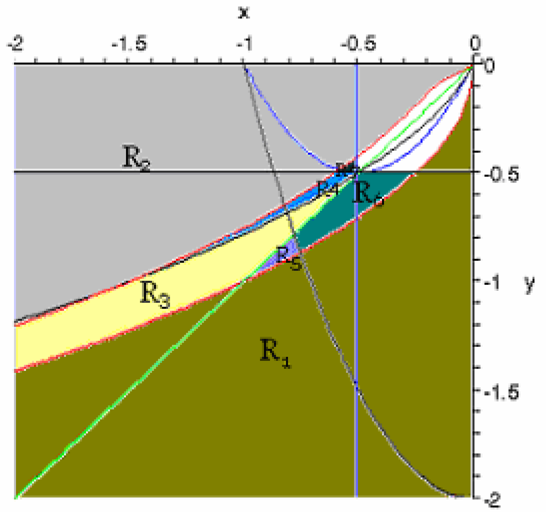
Also , since $x_0 > b\sqrt{-y_0} - y_0^2$,we get $\dfrac{x_0 + y_0^2}{b} < \sqrt{-y_0}$,thus $-y_{-1} < \sqrt{-y_0}$ so $y_{-1}^2 < -y_0$,that is $x_{-1} < -y_{-1}^2$. (4.30)

Hence, from (4.27), (4.28), (4.29) and (4.30) R_7 maps into R_6, by lemma (4-13) R_6 maps into $R_2 \cup R_7$, so there is no point $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ in R_7 such that $\mathrm{H}^{-2n-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in R_7$.That is, there is no point $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ in R_7 such that $\mathrm{H}^{-2n-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ hence there are no periodic points of odd period in R_7.To show that there are no periodic points of even period in R_7.Let P=$\{\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in R_7 : \mathrm{H}^{-2n-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in R_6, \mathrm{H}^{-2n}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in R_7\}$ $\forall n \in N$,if $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in$ P,we claim that the sequence $\langle y_{-2n} \rangle$ is strictly increasing ,since $y_0 > -1-b$,we have $y_0(-1-b) < (1+b)^2$. (4.31)

Also $y_0^2 < (1+b)^2$, $-x_0 > 1+b$,so $-x_0 - y_0^2 > -b(1+b)$, hence $\dfrac{-x_0 - y_0^2}{b} < -1-b$ that is ,$(\dfrac{-x_0 - y_0^2}{b})^2 > (1+b)^2$. (4.32)

Now from (4.31),(4.32), we get that $(\dfrac{-x_0 - y_0^2}{b})^2 > y_0(-1-b) = -y_0 - by_0$ so ,$by_0 > -y_0 - y_{-1}^2$, that is $y_0 < \dfrac{-x_{-1} - y_{-1}^2}{b} = y_{-2}$.In general, since $\begin{pmatrix} x_{-2n} \\ y_{-2n} \end{pmatrix} \in R_7$,we have, $y_{-2n} > -1-b$, $x_{-2n} < -1-b$, so $y_{-2n}(-1-b) < (1+b)^2$, $y_{-2n}^2 < (1+b)^2$.Hence $-x_{-2n} - y_{-2n}^2 > -b(1+b)$, that is $(\dfrac{-x_{-2n} - y_{-2n}^2}{b})^2 > (1+b)^2 > (-1-b)y_{-2n}$.So $y_{-2n-1}^2 > (-1-b)y_{-2n} = -y_{-2n} - by_{-2n} = -x_{-2n-1} - by_{-2n}$, that is , we get that $by_{-2n} > -x_{-2n-1} - y_{-2n-1}^2$, hence $y_{-2n} < \dfrac{-x_{-2n-1} - y_{-2n-1}^2}{b} = y_{-2n-2}$, for all $n \geq 1$.

So, $\langle y_{-2n} \rangle$ is strictly increasing sequence in R_7, so there are no periodic points of even period in P. □



Fig(8):The region R_7 where b=-0.5

Lemma (4-15)

There are no periodic points in R_6.

Proof: by lemma (4-13), R_6 maps into $R_2 \cup R_7$, there are no periodic points in $R_2 \cup R_7$, so there are no periodic points in R_6. □

Theorem (4-16)

There are no periodic points of $H_{0,b}$ in R^2 other than fixed points.

Proof: If $x_0 > -1-b, y_0 = -1-b$,we have $-x_0{}^2 > -(1+b)^2$, $-by_0 = b+b^2$, hence $-x_0{}^2 - by_0 > b+b^2 - 1 - b^2 - 2b = -1-b$, so $x_1 > -1-b$, $y_1 = x_0 > -1-b$,all points $\begin{pmatrix} x \\ y \end{pmatrix}$ in R_3 satisfiy the equation $y < \frac{-x^2}{b}$,if $y_0 > -1-b, x_0 = -1-b$,we get that $-by_0 > b+b^2$, $x^2 = (1+b)^2$ so $-x^2 - by > -1-b$, that is $x_1 > -1-b, y_1 = x_0 = -1-b$

Hence from theorem (3-3) ,and previous lemmas (4-1),(4-2),…,(4-14),,(4-15) , since a periodic point under forwa rd iteration implies one under backwards iteration, and viceversa . □

Referencess

[1] Devaney ,R.L.An Introduction to Chaotic Dynamical Systems ,Second education ,Addison -Wesley ,1989.

[2] Galloway .Introduction to Non Linear Systems and Chaos. The niversity of sydniy, Applied Mathematics ,Math 2006 . Preprint.

[3] Gulick ,D.Encounters with Chaos .McGraw –Hill,Inc.U.S.A,1992.

[4] Hayes .S and wolf .C. Dynamics of a One –Parameter Family of Henon Maps .Department of Mathematics ,university Wichita ,ks 67260,E-mail address:

c wolf @t math.wichita.edu .

[5] Henon.M. A .Two-Dimensional Mapping with a Strange Attractor. Comm. Math. Phys, 50(1): (1976) 69-77

[6] Ilachinski .A.Land Warfare and complexity Mathematical Background and Technical .preprint.