Calculation Lyapunov Exponents for Types of Local Bifurcation

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Abstract

The bifurcation theory is the mathematical study of how and when the solution to a problem changes from there only being one possible solution, to there being two, which is called a bifurcation. Most commonly used in the mathematical study of dynamical systems, the bifurcation occurs when a small smooth change made to the parameter values (the bifurcation parameters) of a system causes a sudden "qualitative" or topological change in its long term dynamical behavior.

In this work, we will recall one parameter of one dimensional vector field to undergo a saddle node bifurcation, transcritical bifurcation and pitchfork bifurcation. Also, one parameter of two dimensional vector fields to undergo a Hopf bifurcation.

Lyapunov exponents measure the rate at which nearby orbits converge or diverge. There are as many Lyapunov exponents as there are dimensions in the state space of the system, but the largest is usually the most important. The goal of our work is to calculate Lyapunov exponent to types of local bifurcation by Mathlab program .We get the saddle node bifurcation has positive Lyapunov exponent if $\mu \le -1$ *,for all the domain. Also, the transcritical bifurcation has positive Lyapunov exponent if* $\mu \le -1$ *,for all the domain.But,the pitchfork bifurcation has negative Lyapunov exponent, for all* $\mu \in R$ *,for all the domain. The last bifurcation is the Hopf bifurcation has positive Lyapunov exponent at (0,0) if* $\mu > 0$ *,but, otherwise the Hopf bifurcation t has negative Lyapunov exponents.*

1-Introduction

The basic goal of the dynamical systems is to understand the eventual or asymptotic behavior of an iterative process. If this process is differential equations whose independent variable is time, then the theory attempts to predict the ultimate behavior of solutions of the equation in either the distant future ($t\to\infty$) or the distant past ($t\to-\infty$). If this process is a discrete process such as the iteration of a function, then the theory hopes to understand the eventual behavior of the points $x, f(x), f^2(x),..., f^n(x)$ as n becomes large [3]. If the differential equation or map depends on a parameter, the dynamics of this equation or map depends on the parameter as well. If the qualitative dynamics of differential equation or map changes as the parameter is varied, this change is called bifurcation. In case of bifurcation there is a special parameter value μ_0 for which the following holds: the dynamics for near to but smaller than μ_0 is qualitatively different from the dynamics for near to but lager than μ_0. This value μ_0 is called a bifurcation value. [2]

The bifurcation occurs in both continuous systems (described by ordinary differential equations) [1,5,7,8,10] and discrete systems (described by maps)[3,6,9,10,11,12].

The bifurcation divided into two principal classes: local bifurcations and global bifurcations. Local bifurcations, which can be analyzed entirely through changes in the local stability properties of equilibria, periodic orbit or other invariant sets as parameters cross through critical thresholds such as saddle node, transcritical, pitchfork, period-doubling (flip), Hopf and Neimark (secondary Hopf) bifurcation. Global bifurcations occur when larger invariant sets, such as periodic orbits, collide with equilibria. This causes changes in the topology of the trajectories in phase space which cannot be confined to a small neighborhood, as is the case with Local bifurcations. Infact, the changes in topology extend out of an arbitrarily large distance (hence "global"). Such as homoclinic in which a limit cycle collides with a saddle point, and heteroclinic bifurcation in which a limit cycle collides with two or more saddle points, and infinite-periodic bifurcation in which a stable node and saddle point simultaneously occur on a limit cycle [8].These bifurcations happen when one varies a single parameter; such bifurcations are called codimension one [10].

From the important dynamical systems is the chaos. A dynamical system is chaotic on a given invariant set Λ for a flow φ when it satisfies certain properties. Of course, Λ could be a very small set in the phase space. And then the assertion of chaos on Λ would not necessarily be of much practical importance. [8]

Many authors believe that the key element of deterministic chaos is the sensitive dependence on the initial conditions. But other authors consider the positive Lyapunov exponent definition of chaos.

The rate of divergence is measured by the largest Lyapunov exponent. A quantitative measure of the sensitive dependence on the initial conditions is the Lyapunov exponent. Aleksandr Mikhailovich Lyapunov was born June 6, 1857 in Yaroslavm, Russia and died November 3, 1918 in Odessa, Russia. He was a Russian mathematician, mechanician and physicist. He left a collection of papers on the equilibrium shape of rotating liquids, on probability, and on the stability of low dimensional dynamical systems. It was from his dissertation that the notion of

Lyapunov exponent emerged. Lyapunov was interested in showing how to discover if a solution to a dynamical system is stable or not for all times [4]. It is the averaged rate of divergence (or convergence) of two neighboring trajectories. Their number is equal to the dimension of the phase space. When speaking about the Lyapunov exponent, the largest one is meant.

Lyapunov exponents measure the rate at which nearby orbits converge or diverge. There are as many Lyapunov exponents as there are dimensions in the state space of the system, but the largest is usually the most important. Roughly speaking the (maximal) Lyapunov exponent is the time constant, lambda, in the expression for the distance between two nearby orbits, exp(lambda*t). If lambda is negative, then the orbits converge in time, and the dynamical system is insensitive to initial conditions. However, if lambda is positive, then the distance between nearby orbits grows exponentially in time, and the system exhibits sensitive dependence on initial conditions.

In this work, we use $\gamma = \sup \frac{1}{t} \log|\Phi(t)|$ to calculate Lyapunov exponent for the one dimensional vector field, and $\gamma_{max}(T) = \frac{1}{T} \ln \frac{|v(T)|}{v_0}$ to calculate Lyapunov exponent for the two or three dimensional vector field.

we explained the Lyapunov exponent definition and we calculate Lyapunov exponent to the four types of the bifurcation.,for nonlinear systems which posses three dimension autonomous differential equation as a model of epidemic, and also with quadratics of six or fewer dimension.

2- Preliminaries

In this section,we recall some definitions which use in this research.

Definition (2.1) [8]

Suppose that the vector field

$\dot{x} = f(x), \; x \in M^n$ ....$(2-1)$

Where M is an n-dimensional phase space. Let $\varphi_t(x)$ be an orbit of $(2-1)$, and let $\Phi(t,x)$ be the fundamental matrix solution of linearized vector field $(2-1)$, and let v be a tangent vector in $T_x M$, the Lyapunov exponent defined as the supremum limit:

$$\gamma(x,v) = \lim_{t\to\infty} \sup \frac{1}{t} \log\left|\Phi(t,x)v\right| \qquad \text{..}(2-2)\,.$$

Since this limit will occur often below, it is nice to give it a more compressed notation. For any function $f(t),$ define the characteristic exponent of f by

$$\chi(f) = \lim_{t\to\infty} \sup \frac{1}{t} \left|f(t)\right| \qquad \text{..}(2-3)$$

That is, $\gamma(x,v) = \chi(\Phi(t,x)v).$

We will use the equation (2-2) in the definition (2.1) to calculate Lyapunov exponent of saddle node, transcritical and pitchfork bifurcation,

Remark (2.2) [8]

To compute the maximal Lyapunov exponent of a system or ordinary differential equations we must integrate both the original system and its linearization $\dot{v} = A(t)v.$ essentially any initial vector v_0 can be used because almost all vectors will have some component along the direction of the maximal Lyapunov direction. We can not compute the limit in $\gamma(x,v) = \lim_{t\to\infty} \sup \frac{1}{t} \left|\Phi(t,x)v\right|$ but instead simply integrate for some long time T and estimate

$$\gamma_{max}(T) = \frac{1}{T} \ln \frac{\left|v(T)\right|}{\left|v_0\right|} \qquad \text{..}(2-4)$$

This quantity will rapidly converge to the maximal exponent; to estimate the error in the computation, it is useful to plot γ_{max} as a function of T.

Definition (2.3) [4]

A system is regular (Lyapunov) if the time average of the trace has a finite limit and equality holds in $\sum_{i=1}^{n} \gamma_i \geq \lim_{t\to\infty} \sup \frac{1}{t} \int_0^t trDf(\varphi_s(x))ds\,.$

Example (2.4)

Consider the Lorenz system

$$\dot{x} = \delta(y - x)$$

$$\dot{y} = rx - y - xz$$

$\dot{z} = xy - bz$

Where the parameters r, δ and b are positive. The linearized equations for a vector $v \in T_x R^3$ are

$$\dot{v} = \begin{bmatrix} -\delta & \delta & 0 \\ r - z & -1 & x \\ y & x & -b \end{bmatrix} v \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2-5)$$

To integrate these equations, we must simultaneously integrate the Lorenz system itself; a simple algorithm to do this and to compute γ_{max} is given in [8]. A plot of the time behavior of $\gamma_{max}(T)$ for two values of r.

(i) If $t = 1000, \delta = 10, b = \frac{8}{3}$ and $r = 28$, then $\gamma_{max}(T) = 0.88$

(ii) If $t = 10000, \delta = 10, b = \frac{8}{3}$ and $r = 28$, then $\gamma_{max}(T) = 0.90$

(iii) $t = 10000, \delta = 10, b = \frac{8}{3}$ and $r = 23$, then $\gamma_{max}(T) = -0.05$

Even though only the largest Lyapunov exponent for the Lorenz system was computed, $\delta \le \sum_{i=1}^{n} \gamma_i$ can be used to estimate the other two exponents, the trace of the Jacobian matrix $(2-5)$ is constant, so that $\delta = tr(Df) = -1 - \delta - b$. Since one exponent vanishes, $\gamma_2 = 0$, for the standard parameters $\gamma_3 \ge \delta - \gamma_1 = -13.66 - \gamma_1$. If the Lorenz were known to be regular, then the supremum limits could be replaced by ordinary limits and the inequality in $\delta \le \sum_{i=1}^{n} \gamma_i$ would become an equality. Therefore,

$\gamma_3 = -13.66 - 0.90 = -14.6$

also we will use the equation (2-4) in remark (2.2) to calculate Lyapunov exponent of Hopf bifurcation. By the term "local" we mean bifurcations occurring in neighborhood of a critical point.

3-Compute Lyapunov Exponents

In this section, we calculate Lyapunov exponents for four generic types of local bifurcation which are saddle node, transcritical, pitchfork and Hopf bifurcation .

1. Saddle node bifurcation.

The normal form for saddle node bifurcation written as follows:

$$\dot{x} = f(x,\mu) = \mu \pm x^2,\ \ x \in R, \mu \in R \ \ ....(3-1)$$

If $\mu \le -1,$ then the vector field (3-1) has positive Lyapunov exponent for all $x \in R,$ otherwise it has negative Lyapunov exponent.

T	μ	x	γ
1000	-1	0	0.0012
1000	-1	0.1	0.0015
1000	-1	-0.1	0.0011
1000	-1	1	0.0019
1000	-1	-1	7.0042
1000	0.1	0	$-\infty$
1000	-0.1	0	-8.2382
1000	-0.1	0.1	-9.3600
1000	-1.1	0	3.3838

Therefore, the saddle node bifurcation has positive Lyapunov exponent if $\mu \le -1.$

2. Transcritical bifurcation.

The normal form for transcritical bifurcation written as follows:

$$\dot{x} = f(x,\mu) = \mu x \pm x^2,\ \ x \in R, \mu \in R \ \ ....(3-2)$$

Too, if $\mu \le -1,$ then the vector field (3-2) has positive Lyapunov exponent for all $x \in R,$ otherwise it has negative Lyapunov exponent.We conclusion that the Lyapunov exponent of saddle node bifurcation is an equal to the Lyapunov exponent of transcritical bifurcation. Therefore, the transcritical bifurcation has positive Lyapunov exponent if $\mu \le -1.$

3. Pitchfork bifurcation.

The normal form for pitchfork bifurcation written as follows:

$$\dot{x} = f(x,\mu) = \mu x \pm x^3,\ \ x \in R, \mu \in R \ \ ....(3-3)$$

The Lyapunov exponent of (3-3) is negative for all $\mu \in R, x \in R$. Therefore, the pitchfork bifurcation has negative Lyapunov exponent

4. The Hopf bifurcation.

The normal form for Hopf bifurcation was found to be:

$$\dot{x} = \alpha(\mu)x - \omega(\mu)y + [a(\mu)x - b(\mu)y](x^2 + y^2) + O(|x|^5, |y|^5)$$

$$\dot{y} = \omega(\mu)x + \alpha(\mu)y + [b(\mu)x + a(\mu)y](x^2 + y^2) + O(|x|^5, |y|^5)$$

For example:

$$\left.\begin{aligned} \dot{x} &= \mu x - y - x(x^2 + y^2) \\ \dot{y} &= x + \mu y - y(x^2 + y^2) \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3-4)$$

The system (3-4) has only positive Lyapunov exponent at the point $(x, y) = (0,0)$ when $\mu > 0$, otherwise it has negative Lyapunov exponent.

T	μ	(x, y)	γ
1000	1	(0,0)	0.4992
1000	-1	(0,0)	-0.4990
1000	0	(0,0)	-2.6191
1000	1	(1,1)	-0.1193
1000	-1	(1,1)	-0.5007
1000	0	(1,1)	-0.0058
1000	1.1	(1,1)	-0.1298
1000	1.1	(0,0)	0.5494
1000	0.1	(0,0)	0.0499
1000	0.1	(0.1,0.1)	-0.0168
1000	-0.1	(0,0)	-0.0499

Therefore, the Hopf bifurcation has positive Lyapunov exponent at (0,0) when $\mu > 0$.

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