Iteration with Periodic Point of Henon Map where $a > \frac{-(1+|b|)^2}{4}$

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Abstract

We study the dynamics of the two dimensional mapping the non linear mapping $H_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a - by - x^2 \\ x \end{pmatrix}$.We study the iteration of all points in the plane . We determine the regain which contain the set of periodic points

1- Introducation

About 30 years ago the French astronomer – mathematician Michel-Henon was searched for a simple two-dimensional map possessing special properties of more complicated systems .The result was a family of maps denoted by $H_{a,b}$ given by

$$H_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 - ax^2 + y \\ bx \end{pmatrix}$$

where a, b are real numbers .These maps defined in above are called Henon maps [5].

we will prove one theorem about iteration of $H_{a,b}$ where $b > 0$ and $a > \frac{-(1+b)^2}{4}$ which is given in [3] . prove we prove that for a closed region

$S_{a,b}=\{\begin{pmatrix}x\\y\end{pmatrix}: |x|\le C_{a,b}, |y|\le C_{a,b}\}$ if $a>\frac{-(1+b)^2}{4}$ and $b>0$ then for all

$\begin{pmatrix}x\\y\end{pmatrix}\in R^2 - S_{a,b}$

either $|x_n|\longrightarrow\infty$ as $n\longrightarrow\infty$ or $|y_{-n}|\longrightarrow\infty$ as $n\longrightarrow\infty$, where $b>0$ and $a<\frac{-(1+b)^2}{4}$, by finding some regions and prove some necessary lemma for our proof..

2- The Set of Periodic Points

The main purpose of this section is to prove one theorem on Henon map $H_{a,b}$ where $a>\frac{-(1+|b|)^2}{4}$. To prove this theorem, we need to prove some lemmas. To state and prove our lemmas, we fix b and define two crucial $a-$values

1- $a_0(b)=\frac{-(1+|b|)^2}{4}$

2- $a_1(b)=2(1+|b|)^2$

and, for any particular $a-$value ,we define $C=C_{a,b}$ by

$C_{a,b}=\frac{1+|b|+\sqrt{(1+|b|)^2+4a}}{2}$, we will go to prove necessary lemmas.

<u>Lemma (2.1)[4]</u>

(i) C is positive real and the larger root of $C^2-(1+|b|)C-a=0$ if and only if $a_0 \le a$.

(ii) $a-|b|C > C$ if and only if $a > a_1$

proof: (i) Since $a_0 \le a$, we have $(1+|b|)^2+4a \ge 0$,so C is positive real number, if $C^2-(1+|b|)C-a=0$,since $(1+|b|)^2+4a \ge 0$.We have two distinct real roots which are $\frac{1+|b|+\sqrt{(1+|b|)^2+4a}}{2}$ and $\frac{1+|b|-\sqrt{(1+|b|)^2+4a}}{2}$,the first is positive and the second may be negative so C is larger root .

(ii)If $a-|b|C > C$,this implies that $a > (1+|b|)C$ (2.1) We have $2C-(1+|b|)=\sqrt{(1+|b|)^2+4a}$.

(2.2) Now let $(1+|b|)=\beta$, then $2C-\beta=\sqrt{\beta^2+4a}$, that is $(2C-\beta)^2=\beta^2+4a$, so $C^2-\beta C=a$.We put the value of a in (2.1) ,we get that $C^2-\beta C > \beta C$ (2.3)

so $C^2 > 2\beta C$,since C is positive we have $C > 2\beta$, that is $C > 2(1+|b|)$. (2.4)

Now by (2.4) $(1+|b|)+\sqrt{(1+|b|)^2+4a} > 4(1+|b|)$,that is $4a+(1+|b|)^2 > 9(1+|b|)^2$, hence $a > 2(1+|b|)^2$.

By the same way, we can prove that if $a > a_1$ then $a-|b|C > C$. □

<u>Lemma (2.2)[4]</u>

(i) The image under $H_{a,b}$ of the horizontal strip $|y_0| \le \gamma$ is the region bounded by the two parabolas $a - |b|\gamma - y_1^2 \le x_1 \le a + |b|\gamma - y_1^2$ and the image under $H_{a,b}$ of the vertical strip $|x_0| \le \gamma$ is the horizontal strip $|y_1| \le \gamma$, where γ is positive real number .

(ii) The inverse image of the vertical strip $|x_0| \le \gamma$ is the region bounded by two parabolas $-\gamma + a - x_{-1}^2 \le by_{-1} \le a + \gamma - x_{-1}^2$ and the image of the horizontal strip $|y_0| \le \gamma$ is the vertical strip $|x_{-1}| \le \gamma$.

Proof: (i) we have $|y_0| \le \gamma$, so $-\gamma \le y_0 \le \gamma$, if $-1 < b < 0$ then $b\gamma \le -by_0 \le -b\gamma$ and $|b| = -b$, hence $-|b|\gamma \le -by_0 \le |b|\gamma$. (2.5)

If $0 < b < 1$,then $b\gamma \ge -by_0 \ge -b\gamma$, $|b| = b$ hence $-|b|\gamma \le -by_0 \le |b|\gamma$. (2.6)

Now in all case $-|b|\gamma \le -by_0 \le |b|\gamma$,so $a - |b|\gamma \le a - by_0 \le a + |b|\gamma$,since $x_0 = y_1$

$a - |b|\gamma - y_1^2 \le x_1 \le a + |b|\gamma - y_1^2$. (2.7)

Also if $|x_0| \le \gamma$, since $x_0 = y_1$,we have $|y_1| \le \gamma$ that means the image under $H_{a,b}$ of the vertical strip $|x_0| \le \gamma$ is the horizontal strip $|y_1| \le \gamma$.

(ii) We have $|x_0| \le \gamma$ so $\gamma \le -x_0 \le -\gamma$ so $a-\gamma-x_{-1}{}^2 \le by_{-1} \le a+\gamma-x_{-1}{}^2, y_0 = x_{-1}$

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so $|y_0| = |x_{-1}|$,the image under $H_{a,b}{}^{-1}$ of the horizontal strip $|y_0| \le \gamma$ is the vertical strip $|x_{-1}| \le \gamma$. □

Lemma (2.3)[2]

Let $P\begin{pmatrix} a \\ b \\ \gamma \end{pmatrix}=\{\begin{pmatrix} x \\ y \end{pmatrix}: a-|b|\gamma-y^2 \le x \le a+|b|\gamma-y^2\}$ $S_h(\alpha,\beta)=R\times[\alpha,\beta]$ and $S_v(\alpha,\beta)=[\alpha,\beta]\times R, \gamma\in R$ and then for the Henon map $H_{a,b}$ the following are hold :

(i) $H_{a,b}(S_v(-\gamma,\gamma))= S_h(-\gamma,\gamma)$.

(ii) $H_{a,b}(S_h(-\gamma,\gamma))\subset P\begin{pmatrix} a \\ b \\ \gamma \end{pmatrix}$.

(iii) $H_{a,b}{}^{-1}(P\begin{pmatrix} a \\ b \\ \gamma \end{pmatrix})\subset S_h(-\gamma,\gamma)$.

Proof: (i) Let $\begin{pmatrix} x \\ y \end{pmatrix}\in H_{a,b}(S_v(-\gamma,\gamma))$, so there exists $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\in S_v(-\gamma,\gamma)$ such that $H_{a,b}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}=\begin{pmatrix} x \\ y \end{pmatrix}$, that is $x = a-by_0-x_0{}^2, y = x_0$,since $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\in[-\gamma,\gamma]\times R$ we have $x_0 \in[-\gamma,\gamma]$,hence $\begin{pmatrix} x \\ y \end{pmatrix}\in R\times[-\gamma,\gamma]$,that is $\begin{pmatrix} x \\ y \end{pmatrix}\in S_h(-\gamma,\gamma)$.

Conversely: Let $\begin{pmatrix} x \\ y \end{pmatrix} \in$ $S_h(-\gamma,\gamma)$, then $x \in$R and $y \in [-\gamma,\gamma]$,so $\frac{a-x-y^2}{b} \in$R ,thus $\begin{pmatrix} y \\ \frac{a-x-y^2}{b} \end{pmatrix} \in [-\gamma,\gamma] \times$R , since $\begin{pmatrix} x \\ y \end{pmatrix}$=$H_{a,b}$ $\begin{pmatrix} y \\ \frac{a-x-y^2}{b} \end{pmatrix}$,we have $\begin{pmatrix} x \\ y \end{pmatrix} \in$ $H_{a,b}(S_v(-\gamma,\gamma)$).

(ii) Let$\begin{pmatrix} x \\ y \end{pmatrix} \in H_{a,b}(S_h(-\gamma,\gamma))$,so there exists $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in S_h(-\gamma,\gamma)$ such that $H_{a,b}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$=$\begin{pmatrix} x \\ y \end{pmatrix}$and $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in$R$\times[-\gamma,\gamma]$,so we get that $|y_0| \le \gamma$ and by lemma (2.2)

Part (i)$a-|b|\gamma - y_1^2 \le x_1 \le a+|b|\gamma - y_1^2$ so $\begin{pmatrix} x \\ y \end{pmatrix} \in$ P $\begin{pmatrix} a \\ b \\ \gamma \end{pmatrix}$.

(iii) Let$\begin{pmatrix} x \\ y \end{pmatrix} \in$ $H_{a,b}^{-1}$(P $\begin{pmatrix} a \\ b \\ \gamma \end{pmatrix}$) ,so there exists $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in$ P $\begin{pmatrix} a \\ b \\ \gamma \end{pmatrix}$ such that

$H_{a,b}^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$=$\begin{pmatrix} x \\ y \end{pmatrix}$, hence $a-|b|\gamma - y_0^2 \le x_0 \le a+|b|\gamma - y_0^2$. Clearly $x \in$R.To show that $y \in [-\gamma,\gamma]$, suppose that $y > \gamma$ then $\frac{a-x_0-y_0^2}{b} > \gamma$
(2.8)
Now if $b > 0$, then from (2.8) $a-x_0-y_0^2 > |b|\gamma$, so $x_0 < a-|b|\gamma - y_0^2$ which is contradiction ,if $b<0$ then from (2.8) $a-x_0-y_0^2 < b\gamma$, so $x_0 > a+|b|\gamma - y_0^2$ which is contradiction .
To show that$y \ge -\gamma$, if $b>0$ then $x_0 \le a+|b|\gamma - y_0^2$ so $-b\gamma \le a-x_0-y_0^2$thus $-\gamma \le \frac{a-x_0-y_0^2}{b} = y$, that is $y \ge -\gamma$,if $b<0$then $|b| = -b$, hence $x_0 \ge a+b\gamma - y_0^2$
so $-b\gamma \ge a-x_0-y_0^2$ thus $-\gamma \le \frac{a-x_0-y_0^2}{b} = y$, that is $y \ge -\gamma$ hence$y \in [-\gamma,\gamma]$ $\begin{pmatrix} x \\ y \end{pmatrix} \in S_h(-\gamma,\gamma)$. □

Proposition (2.4)[2]

Let $S_{a,b}$= $\{\begin{pmatrix} x \\ y \end{pmatrix}$: $|x| \le C_{a,b}, |y| \le C_{a,b}\}$ be a closed region in R^2,for Henon map $H_{a,b}$, if $b \neq 0$ then $H_{a,b}(S_{a,b})$= $P\begin{pmatrix} a \\ b \\ C \end{pmatrix} \cap S_h(-C,C)$.

Proof: From definition of $S_{a,b}$, we have $S_{a,b} = S_h(-C,C) \cap S_v(-C,C)$.

So $H_{a,b}(S_{a,b})$= $H_{a,b}(S_h(-C,C)) \cap H_{a,b}(S_v(-C,C))$. (2.9)

Now by lemma (2.3)(ii) $H_{a,b}(S_h(-C,C)) \subset P\begin{pmatrix} a \\ b \\ C \end{pmatrix}$. (2.10)

By lemma (2.3)(iii),since $H_{a,b}$ is diffeomorphism , we get that

$P\begin{pmatrix} a \\ b \\ C \end{pmatrix} \subset H_{a,b}(S_h(-C,C))$. (2.11)

Hence from (2.10) and (2.11), we get that $H_{a,b}(S_h(-C,C))$= $P\begin{pmatrix} a \\ b \\ C \end{pmatrix}$ (2.12)

From lemma (2.3)(i) we have $H_{a,b}(S_v(-C,C))$= $S_h(-C,C)$. (2.13)

Now by (2.9), (2.12) and (2.13) $H_{a,b}(S_{a,b})$= $P\begin{pmatrix} a \\ b \\ C \end{pmatrix} \cap S_h(-C,C)$. □

Remark (2.5)

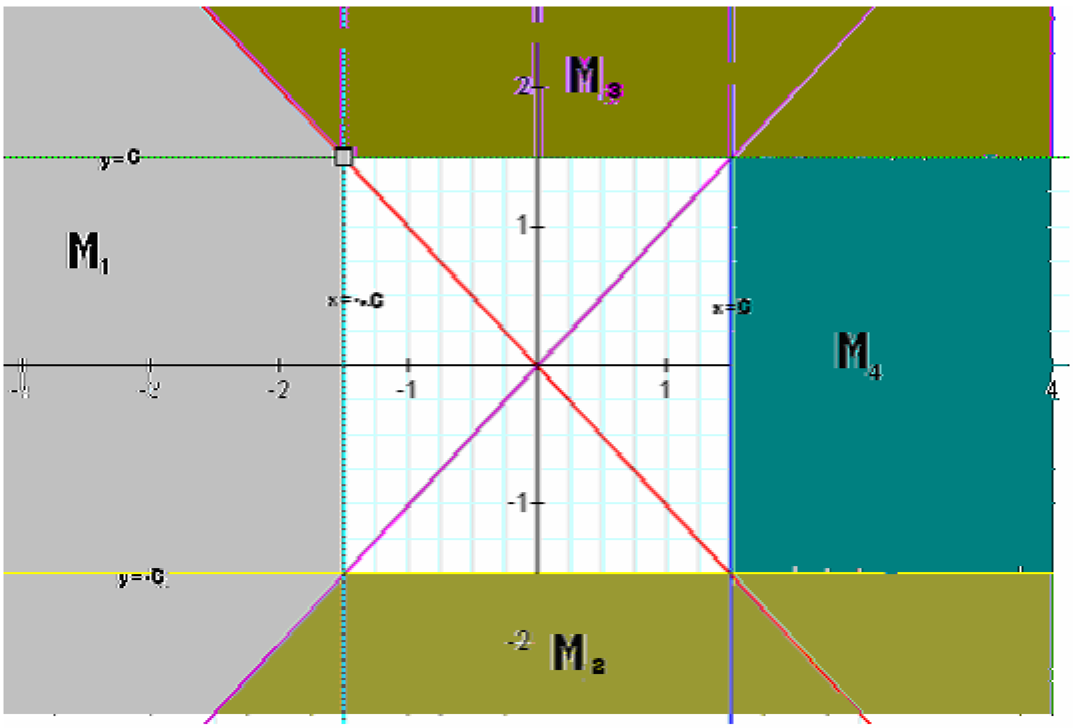
We define some region and proof one theorem on Henon map for $a > a_0$, the regions are M_1, M_2, M_3 and M_4 where :

$M_1 = \{\begin{pmatrix} x \\ y \end{pmatrix} : x < \frac{-\left|C-|y|\right| - C - |y|}{2}\}.$

$M_2 = \{\begin{pmatrix} x \\ y \end{pmatrix} : \ x \geq y, y < -C \}.$

$M_3 = \{\begin{pmatrix} x \\ y \end{pmatrix} : \ x \geq -y, y > C \}.$

$M_4 = \{\begin{pmatrix} x \\ y \end{pmatrix} : \ x > C, |y| < C \}.$



Fig(10)Region M_1, M_2, M_3, M_4

Theorem (2.6)[3]

Suppose $S_{a,b} = \{\begin{pmatrix} x \\ y \end{pmatrix} : \ |x| \leq C_{a,b}, |y| \leq C_{a,b} \}$ is a closed region in R^2.If $a > a_0(b)$ and $b > 0$,then for all $\begin{pmatrix} x \\ y \end{pmatrix} \in R^2 - S_{a,b}$ either $|x_n| \longrightarrow \infty$ as $n \longrightarrow \infty$ or $|y_{-n}| \longrightarrow \infty$ as $n \longrightarrow \infty$.

Lemma (2.7)

Let $H_{a,b}$ be a Henon map, $\begin{pmatrix} x \\ y \end{pmatrix} \in M_1$ and $b > 0$. Then the sequence $\langle x_n \rangle$ is strictly decreasing sequence and $|x_n| \longrightarrow \infty$ as $n \longrightarrow \infty$.

Proof: If $|y_0| \neq C$ then we have two cases:

Case 1: $|y_0| < C$, hence $\left|C - |y_0|\right| = C - |y_0|$, so $x_0 < \dfrac{-(C-|y_0|) - C - |y_0|}{2} = -C$

thus $$x_0 < -C < -|y_0| \qquad .$$

(2.14) Now $x_1 - x_0 = a - by_0 - x_0^2 - x_0$, so $x_1 - x_0 \le a + |b| y_0 - x_0^2 - x_0$.

(2.15)

Since $x_0 < -|y_0|$, we have $|y_0| < -x_0$, so by (2.15) $x_1 - x_0 < a - |b| x_0 - x_0^2 - x_0$

(2.16)

$a - |b| x_0 - x_0^2 - x_0 = a - (|b|+1)x_0 - x_0^2$ but this equation has two roots which are $x_0^{\mp} = \dfrac{-(1+|b|) \mp \sqrt{(1+|b|)^2 + 4a}}{2}$, one of them is $-C$, for any value less than $-C$

$a - (|b|+1)x_0 - x_0^2 < 0$. From (2.14), we have $x_0 < -C$, so $a - (|b|+1)x_0 - x_0^2 < 0$, hence by (2.16) $x_1 - x_0 < 0$, that is $x_1 < x_0$.

Case 2: $|y_0| > C$, hence $\left|C - |y_0|\right| = |y_0| - C$, so $x_0 < \dfrac{-(|y_0| - C) - C - |y_0|}{2} = -|y_0|$

and $|y_0| > C$, thus $x_0 < -|y_0| < -C$. (2.17)

Now $x_1 - x_0 = a - by_0 - x_0^2 - x_0 \le a + |b||y_0| - x_0^2 - x_0$

$< a - |b|x_0 - x_0^2 - x_0 = a - (1+|b|)x_0 - x_0^2$. (2.18)

As case 1, $a - (1+|b|)x_0 - x_0^2 = 0$ has two roots one of them is $-C$.From (2.17) , $x_0 < -C$, so $a - (1+|b|)x_0 - x_0^2 < 0$, hence by (2.18) $x_1 < x_0$.

Now since $x_0 < \frac{-(C - |y_0|) - C - |y_0|}{2}$,we have $x_0 < -C$, that is $y_1 < -C$ so $|y_1| > C$.

Hence $|y_1| - C = |C - |y_1||$. (2.19)

On the other hand $x_1 < x_0$ and x_0 is a negative real number so $x_1 < -|x_0| = -|y_1|$.

From (2.19), we get $x_1 < \frac{-(C - |y_1|) - C - |y_1|}{2}$. As above we get that $x_2 < x_1 < \frac{-(C - |y_1|) - C - |y_1|}{2}$. (2.20)

Now $|y_1| > C$, from (2.20) ,we have $x_1 = y_2 < -C$ so $|y_2| - C = |C - |y_2||$. (2.21)

Also $x_2 < x_1$ and x_1 is a negative real number so $x_2 < -|x_1| = -|y_2|$.

From (2.21) we get $x_2 < \frac{-(C - |y_2|) - C - |y_2|}{2}$.As (2.18), we get $x_3 < x_2$. continuing in this procedure, we get that for a positive integer n $x_n < x_{n-1} < \ldots < x_2 < x_1 < x_0$, that is $\langle x_n \rangle$ is strictly decreasing sequence .

For the second part ,if possible $\langle x_n \rangle$ is bounded, since it is monotone, we get $\langle x_n \rangle$ convergent , since $x_n = y_{n+1}$, that is $\langle x_n \rangle$, also convergent ,

$H_{a,b}$ is continuous. So there exits a fixed point for $H_{a,b}$ in M_1 which is contradiction, hence $\langle x_n \rangle$ is not bounded ,that is $|x_n| \longrightarrow \infty$ as $n \longrightarrow \infty$.□

Remark (2.8)[6]

In theorem (2.7) the equality holds only for $x_0 = -C, y_0 = \mp C$.
Proof: If $b \geq 0$, then $x_1 - x_0 = a - |b| y_0 - x_0^2 - x_0$ if $x_0 = -C$, $y_0 = -C$, then

$x_1 - x_0 = a + (1+|b|)C - C^2$, by lemma (2.1) $x_1 - x_0 = 0$, that is $x_1 = x_0$.

If $b < 0$, then $x_1 - x_0 = a + |b| y_0 - x_0^2 - x_0$ if $x_0 = -C$, $y_0 = C$, then

$x_1 - x_0 = a + (1+|b|)C - C^2$, by lemma (2.1) $x_1 - x_0 = 0$, that is $x_1 = x_0$. □

Lemma (2.9)

Let $H_{a,b}^{-1}$ be the inverse of Henon map , suppose $\begin{pmatrix} x \\ y \end{pmatrix} \in M_2$ and $b > 0$.Then the sequence $\langle y_{-n} \rangle$ is strictly decreasing sequence and $|y_{-n}| \longrightarrow \infty$ as $n \longrightarrow \infty$.

Proof: Let $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in M_2$, from remark (2.5) $x_0 \geq y_0$, $y_0 < -C$, consider

$b(y_{-1} - y_0) = by_{-1} - by_0 = a - x_0 - y_0^2 - by_0$.

(2.22)

Since $x_0 \geq y_0$, we have $a - x_0 - y_0^2 - by_0 \leq a - y_0 - y_0^2 - by_0$

$= a - (1+b)y_0 - y_0^2$.

(2.23)

From (2.22) and (2.23), we get $b(y_{-1}-y_0)\le a-(1+b)y_0-y_0{}^2$. (2.24)

Now the quadratic equation $a-(1+b)y_0-y_0{}^2=0$ has a negative root $y_0^-=-C$

if we take another value $y=y_0^*$ such that $y_0^*<-C$,

becomes $a-(1+b)y-y^2<0$, and we have $y_0<-C$, so $a-(1+b)y_0-y_0{}^2<0$.

From (2.24), we get that $b(y_{-1}-y_0)<0$,since $b>0$, we get $y_{-1}<y_0<-C$ and since $x_{-1}=y_0$,we have $y_{-1}<x_{-1}$ that is $\begin{pmatrix}x_{-1}\\ y_{-1}\end{pmatrix}\in \mathrm{M}_2$.

Now if possible $y_{-k}<y_{-(k-1)}<\ldots<y_{-1}<y_0$, since $y_0<-C$, we have $y_{-k}<-C$.

Also $x_{-k}=y_{-(k-1)}$, we have $y_{-k}<x_{-k}$. (2.25)

$$b(y_{-(k+1)}-y_{-k})\le a-x_{-k}-y_{-k}{}^2-by_{-k}<a-y_{-k}-y_{-k}{}^2-by_{-k}$$
$$=a-(1+b)y_{-k}-y_{-k}{}^2.$$

(2.26)

Now the quadratic equation $a-(1+b)y_{-k}-y_{-k}{}^2=0$ has a negative root $y_{-k}^-=-C$ if we take another value $y=y_{-k}^*$ such that $y_{-k}^*<-C$ becomes $a-(1+b)y-y^2<0$ and we have $y_{-k}<-C$, so $a-(1+b)y_{-k}-y_{-k}{}^2<0$, so from (2.26) we get that $b(y_{-(k+1)}-y_{-k})<0$,since $b>0$, we get $y_{-(k+1)}<y_{-k}$, so we get that $y_{-n}<y_{-(n-1)}<\ldots<y_{-1}<y_0$,that is $\langle y_{-n}\rangle$ is strictly decreasing sequence .

For the second part ,if possible $\langle y_{-n}\rangle$ is bounded, since it is monotone we get $\langle y_{-n}\rangle$convergent ,since $x_{-(n+1)}=y_{-n}$,that is $\langle x_{-n}\rangle$ also convergent , $\mathrm{H}_{a,b}$ is continuous so there exits a fixed point for $\mathrm{H}_{a,b}$ in M_2 which is contradiction hence $\langle y_{-n} \rangle$ not bounded that is $|y_{-n}| \longrightarrow \infty$ as $n \longrightarrow \infty$.□

Lemma(2.10)

The region M_3 maps into M_2 under backward iteration of Henon map $\mathrm{H}_{a,b}$, provided $a > a_0, b > 0$.

Proof: Let $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathrm{M}_3$, hence by remark (2.5) we have $x_0 \geq -y_0$ and $y_0 > C_{a,b}$

so $$b(y_{-1} - y_0) = by_{-1} - by_0 = a - x_0 - y_0^2 - by_0 \quad . \quad (2.27)$$

Since $x_0 \geq -y_0$ and $y_0 > 0$ we have $-by_0 < by_0$ so by (2.27), we have

$$b(y_{-1} - y_0) \leq a + y_0 - y_0^2 + by_0 = a + (1+b)y_0 - y_0^2 \quad . \quad (2.28)$$

Now by lemma (2.1)(i) the quadratic equation $y_0^2 - (1+b)y_0 - a = 0$ has a positive real root $y_0^+ = C$, for any value $y_0^* > C$ this quadratic equation is negative and we have $y_0 > C$ so from (2.28) $b(y_{-1} - y_0) < 0$, since $b > 0$, we get $y_{-1} < y_0$ and since $x_{-1} = y_0$, we have $y_{-1} < x_{-1}$.

Now we have to show that $y_{-1} < -C$ since $y_{-1} + C = \dfrac{a - x_0 - y_0^2}{b} + C$, $y_0 > 0$, we have $b(y_{-1} + C) = a - x_0 - y_0^2 + bC < a + y_0 - y_0^2 + by_0$

$$= a + (1+b)y - y^2 . \quad (2.29)$$

As above and since $y_0 > C$, we have $a + (1+b)y - y^2 < 0$, hence by (2.29) we get that $b(y_{-1} + C) < 0$, since $b > 0$, we get $y_{-1} < -C$, hence $\begin{pmatrix} x_{-1} \\ y_{-1} \end{pmatrix} \in \mathrm{M}_2$.□

Lemma (2.11)

The region M_4 maps into M_1 under forward iteration of the Henon map $H_{a,b}$, provided $a > a_0, b > 0$.

Proof: Let $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in M_4$ hence from remark (2.5), we have $|y_0| < C$ and

$x_0 > C$

from definition of Henon map $x_1 + C = a - by_0 - x_0^2 + C$. (2.30)

Since $x_0 > C$, we have $x_0^2 > C^2$. Furthermore $|y_0| < C$ so $-by_0 < bC$.

Now from (2.30) and (2.31) we get that

$x_1 + C < a + bC - C^2 + C = a + (1+b)C - C^2$

(2.32)

From lemma (2.1)(i) , C is a positive real root $y^2 - (1+b)y - a = 0$

so $a + (1+b)C - C^2 = 0$, from (2.32), we get $x_1 + C < 0$, that is $x_1 < -C$,since $y_1 = x_0, x_0 > C$ and we have $x_1 < -C$, we get $y_1 > x_1$. (2.33)

Now $x_0 + x_1 = x_0 + a - by_0 - x_0^2$. (2.34)

Since $x_0 > C > |y_0|$,so if $y_0 \geq 0$ then $x_0 > y_0$,hence $bx_0 > by_0 > -by_0$.

If $y_0 < 0$ then $x_0 > -y_0$ that is $bx_0 > -by_0$.

From (2.34), we get $x_0 + x_1 < x_0 + a + bx_0 - x_0^2 = a + (1+b)x_0 - x_0^2$. (2.35)

As above C is a positive real root , $x^2-(1+b)x-a=0$,for $x_0>C$ $a+(1+b)x_0-x_0{}^2<0$, from (2.35) ,we get that, $x_0+x_1<0$ that is $x_1<-x_0=-y_1$.

Now we have $x_1<y_1<-x_1$ so $x_1<-|y_1|$, $x_0=|y_1|>C$,which becomes $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \in \mathrm{M}_1$.□

<u>proof of theorem (2.6)</u>

Let $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathrm{R}^2$ - $\mathrm{S}_{a,b}$, by remark (2.5) $\begin{pmatrix} x \\ y \end{pmatrix} \in \bigcup_{i=1}^{4} \mathrm{M}_i$ so we have the following cases :

Case I: If $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathrm{M}_1$, by lemma (2.7) the sequence $\langle x_n \rangle$ is strictly decreasing sequence and $|x_n| \longrightarrow \infty$ as $n \longrightarrow \infty$.

Case II: If $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathrm{M}_2$, by lemma (2.9) the sequence $\langle y_{-n} \rangle$ is strictly decreasing sequence and $|y_{-n}| \longrightarrow \infty$ as $n \longrightarrow \infty$.

Case III: If $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathrm{M}_2$ by lemma (2.10) $\begin{pmatrix} x \\ y \end{pmatrix}$ maps into M_3 under backward iteration of Henon map $\mathrm{H}_{a,b}$ .So from case II the sequence $\langle y_{-n} \rangle$ is strictly decreasing sequence and $|y_{-n}| \longrightarrow \infty$ as $n \longrightarrow \infty$.

Case IV: If $\begin{pmatrix} x \\ y \end{pmatrix} \in M_4$ by lemma (2.11) $\begin{pmatrix} x \\ y \end{pmatrix}$ maps into M_1 under forward iteration of the Henon map $H_{a,b}$ so from case I the sequence $\langle x_n \rangle$ is strictly decreasing sequence and $|x_n| \longrightarrow \infty$ as $n \longrightarrow \infty$. □

Corollary (2.12)[2]

If $a > a_0(b)$ then $P_{er_n}(H_{a,b}) \subset S_{a,b}$.

Proof: Let $\begin{pmatrix} x \\ y \end{pmatrix} \in$ Per($H_{a,b}$),if $\begin{pmatrix} x \\ y \end{pmatrix} \notin S_{a,b}$,then $\begin{pmatrix} x \\ y \end{pmatrix} \in (S_{a,b})^C$,that is $\begin{pmatrix} x \\ y \end{pmatrix} \in R^2$-$S_{a,b}$, by theorem (2.5) either $|x_n| \longrightarrow \infty$ as $n \longrightarrow \infty$, or $|y_{-n}| \longrightarrow \infty$ as $n \longrightarrow \infty$, $x_n = y_{n+1}$, $x_{-(n+1)} = y_{-n}$, so there is no finite orbit for $H_{a,b}$ of $\begin{pmatrix} x \\ y \end{pmatrix}$ which is contradiction .So $\begin{pmatrix} x \\ y \end{pmatrix} \in S_{a,b}$,that is $P_{er_n}(H_{a,b}) \subset S_{a,b}$.□

Definition (2.13)[2]

For $a > a_0(b)$ we define non-escape set of $H_{a,b}$ with respect to a and b by $\Lambda \begin{pmatrix} a \\ b \end{pmatrix} = \{ \begin{pmatrix} x \\ y \end{pmatrix} \in R^2 : \lim_{n \longrightarrow \pm\infty} \left\| H_{a,b}{}^n \begin{pmatrix} x \\ y \end{pmatrix} \right\| \longrightarrow \infty \}^C$.

Corollary (2.14)[2]

For the Henon map $H_{a,b}$, $\Lambda \begin{pmatrix} a \\ b \end{pmatrix} = \bigcap_{m \in Z} H_{a,b}{}^m (S_{a,b})$.

Proof: Let $\begin{pmatrix} x \\ y \end{pmatrix} \in \Lambda \begin{pmatrix} a \\ b \end{pmatrix}$.To show that $\begin{pmatrix} x \\ y \end{pmatrix} \in \bigcap_{m \in Z} H_{a,b}{}^m (S_{a,b})$, if we suppose that

$\begin{pmatrix} x \\ y \end{pmatrix} \notin \bigcap_{m\in Z} H_{a,b}{}^{m}(S_{a,b})$,then there exists $k \in Z$,such that $\begin{pmatrix} x \\ y \end{pmatrix} \notin H_{a,b}{}^{k}(S_{a,b})$,that

means there exists $r \in Z$ such that $H_{a,b}{}^{r}(S_{a,b}) \notin S_{a,b}$,that is

$H_{a,b}{}^{r}(S_{a,b}) \in (\mathrm{S}_{a,b})^{C}$,so by theorem

(2.5) $\left\| H_{a,b}{}^{p} \begin{pmatrix} x \\ y \end{pmatrix} \right\| \longrightarrow \infty$ as $p \longrightarrow \infty$ so $\begin{pmatrix} x \\ y \end{pmatrix} \in (\Lambda \begin{pmatrix} a \\ b \end{pmatrix})^{C}$,that is $\begin{pmatrix} x \\ y \end{pmatrix} \notin \Lambda \begin{pmatrix} a \\ b \end{pmatrix}$

which is contradiction hence $\begin{pmatrix} x \\ y \end{pmatrix} \in \bigcap_{m\in Z} H_{a,b}{}^{m}(S_{a,b})$.

To show that $\bigcap_{m\in Z} H_{a,b}{}^{m}(S_{a,b}) \subset \Lambda \begin{pmatrix} a \\ b \end{pmatrix}$, let $\begin{pmatrix} x \\ y \end{pmatrix} \in \bigcap_{m\in Z} H_{a,b}{}^{m}(S_{a,b})$ so $\begin{pmatrix} x \\ y \end{pmatrix} \in H_{a,b}{}^{m}(S_{a,b})$ for all m in Z, hence $H_{a,b}{}^{n} \begin{pmatrix} x \\ y \end{pmatrix} \in S_{a,b}$ for all $n \in Z$, that means if n is very large

$H_{a,b}{}^{n} \begin{pmatrix} x \\ y \end{pmatrix} \in S_{a,b}$, where $n \longrightarrow \infty$ or $n \longrightarrow -\infty$ then $\lim_{n \longrightarrow \pm\infty} H_{a,b}{}^{n} \begin{pmatrix} x \\ y \end{pmatrix} \in S_{a,b}$ so $\lim_{n \longrightarrow \pm\infty} \left\| H_{a,b}{}^{n} \begin{pmatrix} x \\ y \end{pmatrix} \right\|$ is real number that is $\begin{pmatrix} x \\ y \end{pmatrix} \in \Lambda \begin{pmatrix} a \\ b \end{pmatrix}$ so $\bigcap_{m\in Z} H_{a,b}{}^{m}(S_{a,b}) \subset \Lambda \begin{pmatrix} a \\ b \end{pmatrix}$,

that is $\Lambda \begin{pmatrix} a \\ b \end{pmatrix} = \bigcap_{m\in Z} H_{a,b}{}^{m}(S_{a,b})$.□

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