On a weak form separation axioms and continuous functions in ω- bitopological spaces

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Abstract

The aim of this paper is to introduced a new types of separation axioms in bitopological spaces which is defined by an ω- open set which is defined by H.Z .Hdeib[3] in 1982 ,many results and relation ships are studied in this paper

Key words: ω- open set, pre-open set ,α –open set ,separation axioms , bitopological space.

1. Introduction:

A bitopological space is a non – empty set X with two topologies τ1, τ2 defined on it or it is the triple(X, τ1, τ2) ,such that (X, τ1),(X, τ2) are two topological spaces defined on X ,AL- Swidi and Asaad M .A.Alhosaini [1] , introduced a new notions on an ij- ω- open set in bitopological spaces in 2011

The concept of pre open sets ,semi open sets, β open sets,α open sets ,and b open sets were introduced by (cf. [2,4,6,7]) and extended by (cf. [9,10]).

Let (X ,τ) be a topological space , $A \subset X$, a point $x \in X$,is called a condensation point if for each $U \in \tau$ with $x \in U$,the set $U \cap A$ is uncountable . A is said to be an ω-closed if it contains all its condensation points ,the complement of ω-closed is said to be ω-open set .equivalently a set W is ω-open if for each $x \in W$,there exist $U \in \tau$ with $x \in U$ and U-W is countable[3].

The family of all ω-open set in(X ,τ) , denoted by τ_ω ,forms a topology on X finner than τ.the ω-closure and ω-interior of a set A will be denoted by $cl_\omega A$ and $int_\omega A$ resp. , are defined by:

$cl_\omega A = \bigcap \{F \subset X / F \text{ is } \omega\text{-closed and } A \subset F\}$

$int_\omega A = \bigcup \{ G \subset X / G \text{ is } \omega\text{-open and } G \subset A\}$

In 2009 [10]T.Noiri ,A, AL- Omari ,M,S,M. Lorain introduced and discussed a new notions called(α-ω-open, pre-ω –open ,and b-ω –open) sets in topological spaces.

1.1. Definition [4,9]

Let (X , $\tau_{1,}\tau_2$) be a bitopological space , A$\subset$x ,A is said to be

(i) ij-p open set if A $\subset$i-int(j-clA).
(ii) ij-s open set if A $\subset$ j-cl(i-intA).
(iii) ij-α open set if A $\subset$i-int(j-cl(i-int A))).
(iv) ij-β open set if A $\subset$j-cl(i-int(j-clA)).
(v) ij-b open set if A $\subset$i-int(j-clA)$\cup$j-cl(i-intA) .
(p-open denotes pre open ,and s-open denotes semi open). i,j$\in$ {1,2}.

2. weak Forms of ω-Open Sets in Bitopological Spaces

2.1.Definition[10]

let(X , $\tau_{1,}\tau_2$) be a bitopological space , A$\subset$x ,A is said to be:

ij- pre ω open ,if A $\subset$i-int (j-cl$_\omega$A).

ij- semi ω open ,if A $\subset$j-cl(i-int$_\omega$A).

ij- α ω open ,if A $\subset$i-int(j-cl$_\omega$(i-int A)).

ij- β ω open ,if A $\subset$j-cl(i-int$_\omega$(j-clA)).

ij- b ω open ,if A $\subset$i-int(j-cl$_\omega$A) $\bigcup$ j-cl(i-int$_\omega$A).

The set ij-pre ω open(ij-semi ω open) will be denoted briefly ij-p ω open (ij-s ω open).

2.2 .Remark [1]

It is clear by definition that in any bitopological space the following hold

(i) every i-open set is ij- pre open , ij-s open, ij-α open, ij-β open and ij-b open set
(ii) every ij- pre open set is ij-β open.
(iii) every ij-α open set is ij-s open.
(iv) every ij- pre open (ij-s open)set is -b open set.
(v) the concept of ij-pre open and (ij-s open) sets are independent.
(vi) the concept of ij-α open and ij-β open sets are independent.

2.3. Lemma[1]

(i) every ij-α ω open set is ij-p ω open .
(ii) every ij-s ω open set is ij-b ω open .
(iii) every ij-s ω open set is ij-β ω open .
(iv) every ij-α ω open set is ij-β ω open .
(v) the sets ij-p ω open and ij-s ω open are independent .

2.4.Definition

For $i \neq j, i, j \in \{1,2\}$, A bitopological space (X, τ1, τ2) is said to be :

1-pairwise semi ωT_0-space if for any point x$\in$X there are two ij –s ω open sets G and H such that $x \in G, x \notin H$,and it is denoted by ps ωT_0-space.

2-pairwise α ωT_0-space if for any point x$\in$X there are two ij –α ω open sets G and H such that $x \in G, x \notin H$, and it is denoted by pα ωT_0-space.

3- pairwise pre ωT_0-space if for any point x$\in$X there are two ij-p ω open sets G and H such that $x \in G, x \notin H$,and it is denoted by pp ωT_0-space.

4-pairwise b ωT_0-space if for any point x$\in$X there are two ij –b ω open sets G and H such that $x \in G, x \notin H$,and it is denoted by pb ωT_0-space.

5-pairwise β ωT_0-space if for any point x$\in$X there are two ij –β ω open sets G and H such that $x \in G, x \notin H$,and it is denoted by pβ ωT_0-space.

2.5. Proposition

Any pα ωT_0 -space is p ωT_0-space.

Proof:

Let X is pα $\omega T_0 \rightarrow$ for any point x$\in$X there are two ij – αω open sets G and H such that $x \in G, x \notin H$, by lemma(2.3) G and H are two ij —-ω open sets , hence the result.

2.6. Proposition

a- any ps ωT_0-space is pb ωT_0-space

b- any ps ωT_0-space is pβ ωT_0-space

c- any pα ωT_0-space is pβ ωT_0-space

proof:

same proof of theorem a bove and lemma (2.3).

2.7.Definition

For $i \neq j, i, j \in \{1,2\}$, A bitopological space (X, τ1, τ2) is said to be :

1- pairwise semi ωT_1-space if for any pair of distinct points x$\neq y \in$X there are two ij-s ω-open sets G and H such that $x \in G, x \notin H$,and $y \in H, y \notin G$ and it is denoted by ps$\omega T1$-space.

2- pair wise $\alpha\omega T_1$-space if for any pair of distinct points x$\neq y \in$X there are two ij-α ω-open sets G and H such that $x \in G, x \notin H$,and $y \in H, y \notin G$ and it is denoted by p$\alpha\omega T_1$-space.

3- pairwise pre ωT_1-space if for any pair of distinct points x$\neq y \in$X there are two ij-p ω-open sets G and H such that $x \in G, x \notin H$,and $y \in H, y \notin G$ and it is denoted by pp$\omega T1$-space.

4-Pair wise $b\omega T_1$-space if for any pair of distinct points x$\neq y \in$X there are two ij-b ω-open sets G and H such that $x \in G, x \notin H$,and $y \in H, y \notin G$ and it is denoted by p$b\omega T_1$-space

5- pair wise β ωT_1-space if for any pair of distinct points x$\neq y \in$X there are two ij-β ω-open sets G and H such that $x \in G, x \notin H$,and $y \in H, y \notin G$ and it is denoted by pβ$\omega T1$-space

2.8.Proposition

Any pα$\omega T1$-space is pp$\omega T1$-space .

Proof:

Let X is pα$\omega T1 \rightarrow$ for any pair of distinct points x$\neq y \in$X there are two ij –α ω open sets G and H such that $x \in G, x \notin H$,and $y \in H, y \notin G$,and by lemma (2.3) G and H are two ij-p ω open
,hence the result.

2.9.Proposition

a- any psω T_1-space is pb$\omega T1$-space
b- any psω T_1 -space is pβ$\omega T1$ -space
c- any pα$\omega T1$ -space is pβ$\omega T1$-space

proof:

same proof of theorem a bove and lemma (2.3).

2.10 Definition

For $i \neq j, i, j \in \{1,2\}$, A bitopological space (X, τ1, τ2) is said to be :

1- pair wise semi ωT_2 -space if for any pair of distinct points x$\neq y \in$X, there are two ij –ω open sets G and H such that $x \in G, y \in H$,and $G \cap H = \phi$ and it is denoted by ps ωT_2 -space .

2- pair wise α ωT_2 -space if for any pair of distinct points x$\neq y \in$X, there are two ij-α ω-open sets G and H such that $x \in G, y \in H$,and $G \cap H = \phi$ and it is denoted by pα ωT_2 -space.

3-pair wise p ωT_2 -space if for any pair of distinct points x$\neq y \in$X, there are two ij-p ω-open sets G and H such that $x \in G, y \in H$,and $G \cap H = \phi$ and it is denoted by pp ωT_2 -space.

4-pair wise b ωT_2 -space if for any pair of distinct points x$\neq y \in$X, there are two ij- b ω-open sets G and H such that $x \in G, y \in H$,and $G \cap H = \phi$ and it is denoted by pb ωT_2 -space.

5-pair wise β ωT_2 -space if for any pair of distinct points x$\neq y \in$X, there are two ij-β ω-open sets G and H such that $x \in G, y \in H$,and $G \cap H = \phi$ and it is denoted by pβ ωT_2 -space

2.11 Proposition

Any pα ωT_2 -space is pp ωT_2

Proof:

Let X is pα $\omega T_2 \rightarrow$ for any pair of distinct points x$\neq y \in$X, there are two ij –α ω open sets G and H such that $x \in G, y \in H$,and $G \cap H = \phi$,by lemma(2.3) G and H are two ij-p ω open sets,and hence the result.

2.12 Proposition

a- any ps ωT_2 -space is pb ωT_2 -space.
b- Any ps ωT_2 -space is pβ ωT_2 -space.
c- Any pα ωT_2 -space is pβ ωT_2 -space.

Proof:
Same proof of theorem above and by using lemma (2.3).

2.13 Proposition

a- any pωT_2-space is p$\omega T1$-space.

b- Any p$\omega T1$-space is pωT_0-space.

Proof:

(a) clear by definitions of pωT_2 and p$\omega T1$

(b) clear by definitions of p$\omega T1$ and pωT_0

2.14 Example

Let X ={a,b,c,d } and $\tau_1 = \tau_2 = \{Ø, X, \{b\}, \{a,b\}, \{b,c\}, \{a,b,c\}\}$.

Then X is pαωT_0 (ppωT_0 , psωT_0, pβωT_0) but it is not p$\alpha\omega T_1$ (pp$\omega T1$, ps$\omega T1$, pβ$\omega T1$) .

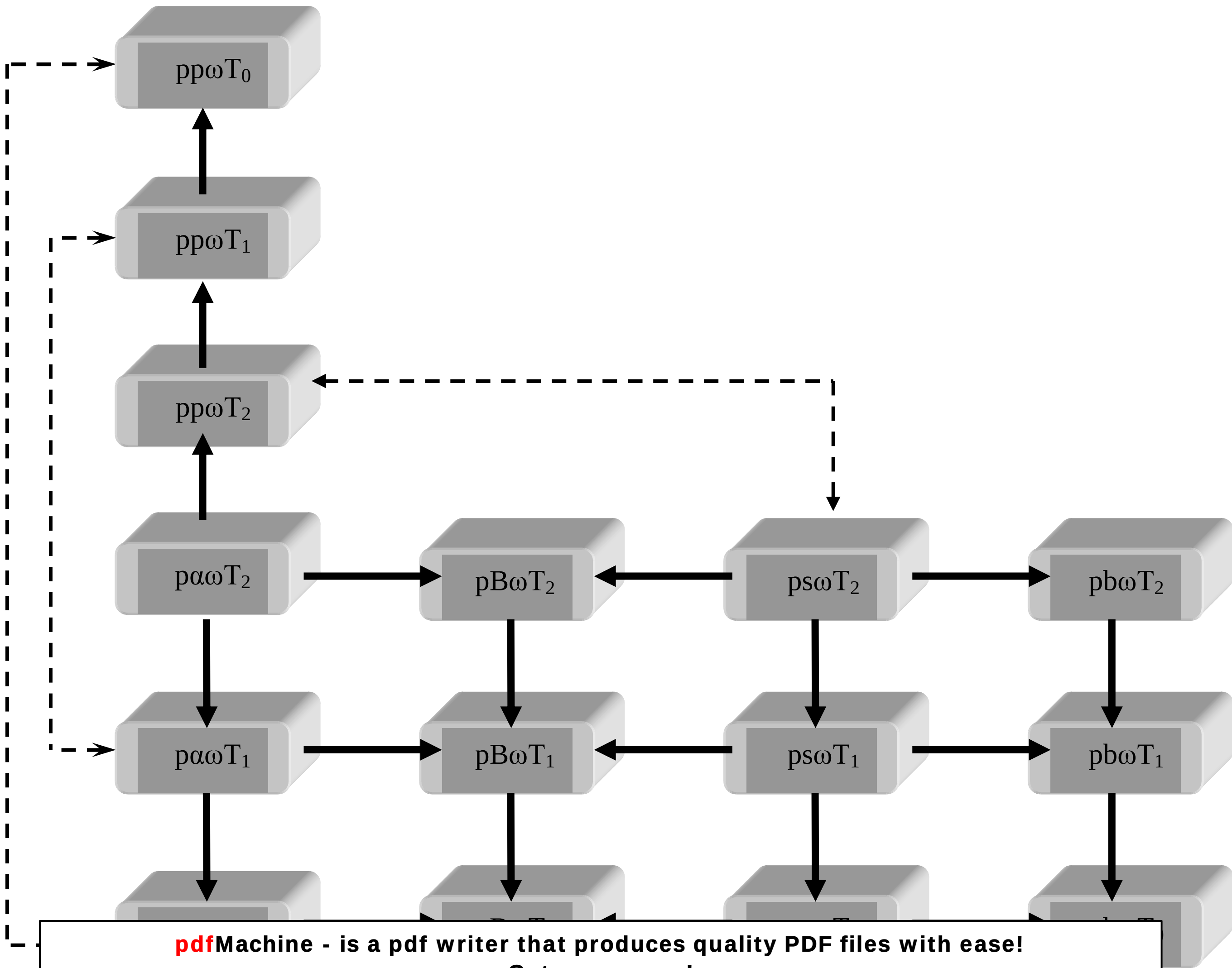
2.15 Example

Let X$\neq$ Ø,and $\tau_1 = \tau_2$ are the excluding point topologies ,then X is pbωT_0 but not p$b\omega T_1$.

2.16 Example

Let X is infinite set and $\tau_1 = \tau_2$ are the cofinite topologies then X is pα$\omega T1$ (pp$\omega T1$, psω T_1, pβ$\omega T1$) but it is not pαωT_2 (ppωT_2, psωT_2, pβωT_2).

The following diagram show us the relation ships between these spaces.



Where $\longrightarrow$ means the arrow is not reversible and
$\leftarrow - - \rightarrow$ Means the arrows is independent.

3. some new types of continuous functions

3.1 Definition:

For $i \neq j, i, j \in \{1,2\}$, a function $f:(X,\tau_1,\tau_2) \to (Y,\rho_1,\rho_2)$ is said to be $ij-s\omega-$(resp. $ij-\alpha\omega, ij-p\omega, ij-b\omega, ij-\beta\omega$) open function if $f(U)$ is $ij-s\omega-$(resp. $ij-\alpha\omega-, ij-p\omega-, ij-b\omega, ij-\beta\omega$) open set in Y for each $ij-s\omega-$(resp. $ij-\alpha\omega, ij-p\omega, ij-b\omega, ij-\beta\omega$) open set U in X.

3.2 Definition:

For $i \neq j, i, j \in \{1,2\}$,a function $f:(X,\tau_1,\tau_2) \to (Y,\rho_1,\rho_2)$ is said to be $ij-s\omega-$(resp. $ij-\alpha\omega, ij-p\omega, ij-b\omega, ij-\beta\omega$) continuous function if $f^{-1}(U)$ is $ij-s\omega-$(resp. $ij-\alpha\omega-, ij-p\omega-, ij-b\omega, ij-\beta\omega$) open set in X for each $ij-s\omega-$(resp. $ij-\alpha\omega, ij-p\omega, ij-b\omega, ij-\beta\omega$) open set U in Y.

3.3 Proposition:

i) any $ij-\beta\omega$ -continuous function is $ij-\alpha\omega$ -continuous
ii) any $ij-s\omega-$ continuous function is $ij-\alpha\omega$ continuous
iii) any $ij-\alpha\omega$ -continuous function is , $ij-p\omega$ -continuous
iv) any $ij-b\omega$ -continuous function is $ij-s\omega-$ continuous

proof :(i)

let $f:(X,\tau_1,\tau_2) \to (Y,\rho_1,\rho_2)$ be a $ij-\beta\omega$ -continuous function, and U is $ij-\beta\omega$ -open set in Y $\Rightarrow$ $f^{-1}(U)$ is $ij-\beta\omega$ -open set in X, since any $ij-\alpha\omega$ -open set is $ij-\beta\omega$ -open set $\Rightarrow U$ and $f^{-1}(U)$ are $ij-\alpha\omega$ -open sets, hence f is $ij-\alpha\omega$ -continuous.

Proof :(ii),(iii) and(iv)

Obvious by definitions and lemma(2.3).

3.4 Remark:

the converse of each one of proposition above need not true, which can be summarized by the following example.

3.5 Example:

Let X={a,b,c}, τ_1={X ,Ø ,{a},{a,b} } , τ_2={ X ,Ø ,{c} } and Y={1,2,3}, ρ_1={ Y, Ø,{1},{1,2}, ρ_2={ Y, Ø,{3} } and $f:(X,\tau_1,\tau_2) \to (Y,\rho_1,\rho_2)$ be a function defined by f (a)= 1, f (b)= f (c)=2 and, then f is $12-\alpha\omega$ -continuous but neither $12-\beta\omega$ -continuous nor $12-s\omega-$ continuous.

The following diagram shows the relationships between these functions.

$$
\begin{array}{ccccc}
ij-\beta\omega\text{-continuous} & \Rightarrow & ij-\alpha\omega\text{ -continuous} & \Rightarrow & ij-p\omega\text{-continuous} \\
 & & \Uparrow & & \\
ij-s\omega-\text{continuous} & \Leftarrow & ij-b\omega\text{ -continuous} & &
\end{array}
$$

3.6 Proposition:

Let $f:(X,\tau_1,\tau_2)\to(Y,\rho_1,\rho_2)$ be a bijective $ij-s\omega-$open function ,then (Y,ρ_1,ρ_2) is ps $\omega T1$-space if (X,τ_1,τ_2) is ps $\omega T1$-space.

Proof:

Let $y_1\neq y_2\in Y$, since f is 1-1 $\Rightarrow$ there exist $x_1\neq x_2\in X$ such that $f(x_1)=y_1, f(x_2)=y_2\in X$,since (X,τ_1,τ_2) is ps $\omega T1$,then there are two ij-s ω-open sets G and H such that $x_1\in G, x_2\notin G$ nd $x_2\in H, x_1\notin H$ $\Rightarrow y_1=f(x_1)\in f(G)$ but $y_2=f(x_2)\notin f(G)$ and $y_2=f(x_2)\in f(H)$ but $y_1=f(x_1)\notin f(H)$, since f is onto $ij-s\omega-$open function, hence $f(H)$, $f(G)$ are $ij-s\omega-$open sets in (Y,ρ_1,ρ_2) ,then (Y,ρ_1,ρ_2) is ps $\omega T1$-space.

3.7 Proposition:

Let $f:(X,\tau_1,\tau_2)\to(Y,\rho_1,\rho_2)$ be a bijective $ij-\alpha\omega$ (resp., $ij-p\omega$, $ij-b\omega$, $ij-\beta\omega$) open function then (Y,ρ_1,ρ_2) is p $\alpha\omega T_1$ (resp. pp $\omega T1$, p $b\omega T_1$, pβ $\omega T1$)if (X,τ_1,τ_2) is p $\alpha\omega T_1$ (resp. pp $\omega T1$, p $b\omega T_1$, pβ $\omega T1$).

Proof:

Same proof of proposition above and by replacing $ij-s\omega-$open set by($ij-\alpha\omega$, $ij-p\omega$, $ij-b\omega$, $ij-\beta\omega$) open set.

3.8 Proposition:

Let $f:(X,\tau_1,\tau_2)\to(Y,\rho_1,\rho_2)$ be a bijective $ij-s\omega-$open function ,then (Y,ρ_1,ρ_2) is ps ωT_2 -space if (X,τ_1,τ_2) is ps ωT_2 -space.

Proof:

Let $y_1\neq y_2\in Y$, since f is 1-1 $\Rightarrow$ there exist $x_1\neq x_2\in X$ such that $f(x_1)=y_1, f(x_2)=y_2\in X$,since (X,τ_1,τ_2) is ps ωT_2 -space ,then there are two ij-s ω-open sets G and H such that $x_1\in G$ and $x_2\in H, G\cap H=\phi\Rightarrow$

$y_1=f(x_1)\in f(G)$ and $y_2=f(x_2)\in f(H)$ and $f(G\cap H)=f(G)\cap f(H)=\phi$ since f is onto $ij-s\omega-$open function ,then $f(G), f(H)$ are $ij-s\omega-$open sets in (Y,ρ_1,ρ_2) ,then (Y,ρ_1,ρ_2) is ps ωT_2 -space.

3.9 Proposition:

Let $f:(X,\tau_1,\tau_2)\to(Y,\rho_1,\rho_2)$ be a bijective $ij-\alpha\omega$ (resp., $ij-p\omega$, $ij-b\omega$, $ij-\beta\omega$) open function then (Y,ρ_1,ρ_2) is p $\alpha\omega T_2$ (resp. pp ωT_2 , p $b\omega T_2$, pβ ωT_2)if (X,τ_1,τ_2) is p $\alpha\omega T_2$ (resp. pp ωT_2 , p $b\omega T_2$, pβ ωT_2).

Proof:

Same proof of proposition above and by replacing $ij-s\omega-$open set by($ij-\alpha\omega$, $ij-p\omega$, $ij-b\omega$, $ij-\beta\omega$) open set.

3.10 Proposition:

Let $f:(X,\tau_1,\tau_2)\to(Y,\rho_1,\rho_2)$ be a 1-1 and $ij-s\omega-$continuous function ,if (Y,ρ_1,ρ_2) is ps ωT_2 -space then (X,τ_1,τ_2) is ps ωT_2 -space.

Proof:

Let $x_1\neq x_2\in X$, since f is 1-1 $\Rightarrow f(x_1)\neq f(x_2)\in Y$,since (Y,ρ_1,ρ_2) is ps ωT_2 -space, then there are two ij-s ω-open sets G and H in X such that $f(x_1)\in G$ and $f(x_2)\in H, G\cap H=\phi\Rightarrow x_1\in f^{-1}(G), x_2\in f^{-1}(H)$, $f^{-1}(G\cap H)=f^{-1}(G)\cap f^{-1}(H)=\phi$,since f is $ij-s\omega-$continuous then $f^{-1}(G), f^{-1}(H)$ are $ij-s\omega-$open sets in (X,τ_1,τ_2) ,hence (X,τ_1,τ_2) is ps ωT_2 -space.

3.11 Proposition:

Let $f:(X,\tau_1,\tau_2)\to(Y,\rho_1,\rho_2)$ be a 1-1, $ij-\alpha\omega$ (resp., $ij-p\omega$, $ij-b\omega$, $ij-\beta\omega$) continuous function then (X,τ_1,τ_2) is p $\alpha\omega T_2$ (resp. pp ωT_2 , p $b\omega T_2$, pβ ωT_2) if (Y,ρ_1,ρ_2) is p $\alpha\omega T_2$ (resp. pp ωT_2 , p $b\omega T_2$, pβ ωT_2).

Proof:

Same proof of proposition above and by replacing $ij-s\omega-$open set by($ij-\alpha\omega$, $ij-p\omega$, $ij-b\omega$, $ij-\beta\omega$) open set.

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