Applications of Elliptic Varational Inequality Methods to The Solution of Some Nonlinear Elliptic Equations

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Abstract

For solving some non-linear elliptic equations it may be convenient, from the theoretical and numerical points of view, to see them as elliptic variational inequalities. We shall consider in this paper of such situation, the family of non-linear elliptic equations. The equivalence between the non-linear elliptic equation and the elliptic variational inequality EVI of the second kind is proved.

الخلاصة

حل بعض المعادلات التناقصية الغير خطية ومن الملائم نظرياً وعددياً معاملتها كمتباينات تقايرية ناقصية. التكافؤ بين المعادلة التناقصية الغير خطية والمتباينة التقايرية الناقصية من النوع الثاني قد تم برهانه.

1. Introduction

The variational inequality is an important and very useful class of non-linear problems arising from mechanics, physics etc. the theory of variational inequalities is rich and exciting, within it, one can find a wealth of powerful ideas which do not only reveal fundamental facts on the qualitative behavior of solutions to important classes of non-linear boundary value problems, but which also provide a natural frame work for a host of relatively new numerical methods . (Lions, 1967).

1.1: Notations

* V: real Hilbert space with scalar product (.,.) and associated norm $\|.\|$.
* V^* : the dual space of V.
* a(.,.) : $V \times V \to \Re$ is a bilinear , continuous and V-elliptic form on $V \times V$.

A bilinear form a(.,.) is said to be V-elliptic if there exists a positive constant α such that $a(v,v) \geq \alpha \|v\|^2 \quad \forall\, v \in V.$

In general we do not assume a(.,.) to be symmetric, since in some applications non-symmetric bilinear forms may occur naturally (Lions, 1967).

* $L: V \to \Re$ continuous, linear functional.
* K is a closed, convex , non-empty subset of V.
* $j(\cdot): V \to \overline{\Re} = \Re \cup \{\infty\}$ is a convex , lower semi- continuous (l.s.c.) and proper functional.

(j(.) is proper if $j(v) > -\infty \quad \forall v \in V$ and $j \neq \infty$).

1.2: EVI of First Kind

To find $u \in V$ such that u is a solution of the problem

$$\begin{cases} a(u,v-u)+j(v)-j(u) \geq L(v-u) \quad , \quad \forall v \in V \\ u \in V \end{cases}$$

2. Theoretical and Numerical Analysis of Some Mildly Non-Linear Elliptic Equations

2.1: Formulation of the continuous problem (Vaunberg, 1973; Asmar, 2000):

Let Ω be a bounded domain of $\Re^N$ (N≥2) with a smooth boundary Γ. We consider

* $V = H_0^1(\Omega)$,
* L(v)=<f,v> , f∈V* = $H^{-1}(\Omega)$,
* $\phi : \Re \to \Re, \quad \phi \in C^{\circ}(\Re)$, non-decreasing function with $\phi(0) = 0$

We then consider the following non-linear elliptic equation (P) defined by (Vainberg, 1973; Asmar, 2000):-

Find u∈V such that

$$(P)\cdots\cdots\begin{cases} a(u,v) + <\phi(u), v> = L(v) & \forall v \in V, \\ \phi(u) \in L^1(\Omega) \cap H^{-1}(\Omega). \end{cases}$$

It follows from the Riesz representation Theorem that there exists A∈ϑ(V,V*) such that a(u,v)=<Au,v> ∀u,v∈V. Therefore (P) is equivalent to

$$\left.\begin{aligned} & Au + \phi(u) = f, \\ & u \in V, \\ & \phi(u) \in L^1(\Omega) \cap H^{-1}(\Omega). \end{aligned}\right\}\cdots\cdots\cdots(2.1)$$

2.2: A variational inequality related to (P).

2.2.1: Definition of the variational inequality

Let $\Phi(t) = \int_0^t \phi(\tau) d\tau$, ……………………(2.2)

$$D(\Phi) = \{v \in V : \Phi(v) \in L^1(\Omega)\}\ldots\ldots\ldots\ldots(2.3)$$

The functional $j : L^2(\Omega) \to \overline{\Re}$ is defined by

$$\left.\begin{aligned} & j(v) = \int_\Omega \Phi(v) dx \quad & if \quad \Phi(v) \in L^1(\Omega), \\ & j(v) = +\infty \quad & if \ \Phi(v) \notin L^1(\Omega) \end{aligned}\right\}\ldots\ldots\ldots(2.4)$$

Instead of studying the problem (P) directly , it is natural to associate to (P) the following EVI of the second kind (Chipot & Michaille, 1987):-

$$(\pi)\ldots\ldots\begin{cases} a(u, v-u) + j(v) - j(u) \geq L(v-u) & \forall v \in V, \\ u \in V \end{cases}$$

If a(.,.) is symmetric, a standard method to study (P) is to consider it as the formal Euler equation of the following minimization problem encountered in the calculus of variations.

$$\left.\begin{aligned} & J(u) \leq J(v) \quad & \forall\, v \in V, \\ & u \in V \end{aligned}\right\}\ldots\ldots\ldots\ldots(2.5)$$

where $J(v) = \frac{1}{2} a(v,v) + \int_\Omega \Phi(v) dx - L(v)$……..(2.6)

2.2.2: Properties of j(.):

Since $\phi : \Re \to \Re$ is non-decreasing and continuous with $\phi(0) = 0$, we have

$\Phi \in C^1(\Re), \Phi$ convex , $\Phi(0)=0$, $\Phi(t)\geq 0 \quad \forall t\in\Re$ ....(2.7)

The properties of j(.) are given by the following lemma:

Lemma 2.1: The functional j(.) is convex , proper and l.s.c. over $L^2(\Omega)$.

Proof: since $j(v)\geq 0 \quad \forall v\in L^2(\Omega)$ it follows that j(.) is proper. The convexity of j(.) is obvious from the fact that Φ is convex.

Let us prove that j(.) is l.s.c. Let $(V_n)_n$, $v_n\in L^2(\Omega)$ be such that

$\lim_{n\to\infty} V_n = v$ strongly in $L^2(\Omega)$.

Then we have to prove that

$\lim_{n\to\infty} \inf j(v_n) \geq j(v)$..(2.8)

If $\lim_{n\to\infty} \inf j(v_n)=+\infty$ the property is proved.

Therefore assume that $\lim_{n\to\infty} \inf j(v_n)=\ell\langle\infty$. Hence we can extract a subsequence $(v_{n_k})_{n_k}$ such that

$\lim_{k\to\infty} j\left(V_{n_k}\right) = \ell$ ....(2.9)

$V_{n_k} \to v$ a.e. in Ω ....(2.10)

since $\Phi \in C^1(\Re)$, (2.10) implies

$\lim_{k\to\infty} \Phi\left(V_{n_k}\right) = \Phi(v)$.................(2.11)

Moreover $\Phi(v) \geq 0$ a.e. and (2.9) implies that $\left\{\Phi\left(V_{n_k}\right)\right\}_k$ is bounded in $L^1(\Omega)$. ....(2.12).

Hence by Fatou's Lemma , from (2.11) and (2.12), we have

$$\left.\begin{aligned}&\Phi(v)\in L^1(\Omega),\\ &\lim_{k\to\infty} inf \int_\Omega \Phi(v_{n_k})dx \geq \int_\Omega \Phi(v)dx\end{aligned}\right\}\text{...........(2.13)}$$

From (2.9) and (2.13) we obtain (2.8)

This proves the lemma.

Corollary 2.1: The functional j(.) restricted to V is convex, proper, l.s.c.

2.2.3: Existence and Uniqueness results for (π) (Chipot & Michaille, 1987)

Theorem 2.1: Under the above hypotheses on V, a(.,.), L(.), Φ(.) the problem (π) has a unique solution in $V\cap D(\Phi)$.

Proof: Since V, a(.,.) , L(.), j(.) have the properties (corollary 2.1) required to apply theorem of existence and uniqueness results for EVI of second kind [1], the EVI of the second kind, (π), has a unique solution u in V.

Let us show that $u\in D(\Phi)$. Taking v=0 in (π) we obtain

$a(u,u) + j(u) \leq L(u) \leq \|f\| \cdot \|u\|_V$.......................(2.14)

since $j(u)\geq 0$, using the ellipticity of a(.,.) we obtain

$$\|u\|_V \le \frac{\|f\|}{\alpha} \text{.....................(2.15)}$$

which implies

$$j(u) \le \frac{\|f\|^2}{\alpha} \text{.....................(2.16)}$$

This implies $u \in D(\Phi)$.

Remark : If a(.,.) is symmetric, (π) is equivalent to (2.5).

2.3: Equivalence between (P) and (π)

In this section we shall prove that (P) and (π) are equivalent. First we prove that the unique solution of (π) is also a solution of (P). In order to prove this result we need to prove that $\phi(u)$ and $u\phi(u)$ belong to $L^1(\Omega)$.

Proposition 2.1: Let u be the solution of (π). Then $u\phi(u)$ and $\phi(u)$ belong to $L^1(\Omega)$.

Proof: Here we use a truncation technique. Let n be a positive integer. Define

$$K_n = \{v \in V : |v(x)| \le n \qquad a.e.\}.$$

Since K_n is a closed, convex , non-empty subset of V, the following variational inequality

$$(\pi_n)\text{..........}\begin{cases} a(u^n, v-u^n) + j(v) - j(u^n) \ge L(v-u^n) & \forall v \in K_n, \\ u^u \in k_n \end{cases}$$

has a unique solution (in order to apply theorem of existence and uniqueness results for EVI of second kind [1], we need to replace j by $j+I_{kn}$ where I_{kn} is the indicator functional of k_n).

$$I_{k_n}(v) = \begin{cases} 0 & if \;\; v \in k_n \\ +\infty & if \;\; v \notin k_n \end{cases}$$

Now we prove that $\lim_{n\to\infty} u_n = u$ weakly in V, where u is the solution of (π).

Since $0 \in k_n$, taking v=0 in (π_n) we obtain as in Theorem 2.1 in the previous section that:

$$\|u_n\|_v \le \frac{\|f\|}{\alpha}, \text{...........................(2.17)}$$

$$j(u_n) \le \frac{\|f\|^2}{\alpha} \text{...........................(2.18)}$$

It follows from (2.17) that there exists a subsequence $\{u_{n_k}\}_{n_k}$ of $(u_n)_n$ and $u^* \in V$ such that

$$\lim u_{n_k} = u^* \text{ weakly in V ....(2.19)}$$

Moreover, from the compactness of the canonical injection from $H_0^1(\Omega)$ to $L^2(\Omega)$ and from (2.19), it follows that

$\lim_{k\to\infty} u_{n_k} = u^*$ strongly in $L^2(\Omega)$. ……(2.20)

Relation (2.20) implies that wecan extract a subsequence, still denoted by $\left(u_{n_k}\right)_{n_k}$, such that

$\lim_{k\to\infty} u_{n_k} = u^*$ a.e. in Ω………………..(2.21)

Now let $v\in V\cap L^\infty(\Omega)$, then, for large k, we have $v \in k_{n_k}$ and

$a(u_{n_k}, u_{n_k}) + j(u_{n_k}) \le a(u_{n_k}, v) + j(v) - L(v - u_{n_k})$……..(2.22)

since $\lim_{k\to\infty} inf\ a(u_{n_k}, u_{n_k}) \ge a(u^*, u^*)$

$\lim_{k\to\infty} inf\ j(u_{n_k}) \ge j(u^*)$ it follows from (2.19) and (2.22) that

$$\begin{cases} \{a(u^*, u^*) + j(u^*) \le a(u^*, v) + j(v) - L(v - u^*) & \forall v \in L^\infty(\Omega) \cap V, \\ u^* \in V \end{cases}$$

which can also be written as

$$\begin{cases} a(u^*, v - u^*) + j(v) - j(u^*) \ge L(v - u^*) & \forall v \in V \cap L^\infty(\Omega), \\ u^* \in V & ,…………..(2.23) \end{cases}$$

For n>0, define τ_n : $V \to k_n$ by

$\tau_n v = Inf(n, sup(-n, v))$ (see Figure 2.1) ………..(2.24)

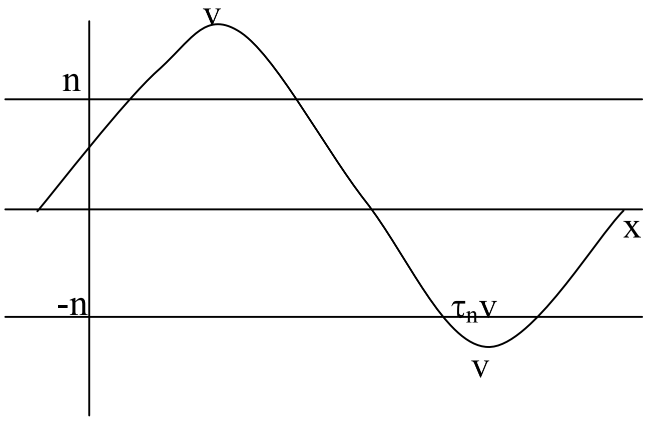

Figure 2.1

Then from the corollary (If V^+ and V^- denote the positive and the negative parts of v for $v\in H^1(\Omega)$ (respectively $H_0^1(\Omega)$ then the map $v\to\{v^+, v^-\}$ is continuous from $H^1(\Omega) \to H^1(\Omega) \times H^1(\Omega)$ (respectively $H_0^1(\Omega) \to H_0^1(\Omega) \times H_0^1(\Omega)$. Also $v \to |v|$ is continuous), [5] we have

$$\left.\begin{aligned}&\lim_{n\to\infty}\tau_n v = v \quad \textit{strongly in } V,\\&\lim_{n\to\infty}\tau_n v = v \quad \textit{a.e. in } \Omega\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(2.25)$$

Moreover , we obviously have,

$|\tau_n v(x)| \le |v(x)|$ a.e. , ………………(2.26)

v(x). τ_n v(x)≥0 a.e. …………………(2.27)

If follows then from (2.25) –(2.27) and from the various properties of that

$\Phi(\tau_n v) \le \Phi(v)$ a.e., …………………(2.28)

$\lim_{n\to\infty}\Phi(\tau_n v) = \Phi(v)$ a. e. ……………………(2.29)

Since $\tau_n v \in L^\infty(\Omega)\cap V$ it follows from (2.23) that

$$\left.\begin{aligned}&a(u^*,\tau_n v - u^*) + j(\tau_n - v) - j(u^*) \ge L(\tau_n v - u^*)\} \qquad \forall v \in V,\\&u^* \in V\end{aligned}\right\}\ldots\ldots(2.30)$$

If v∉D(Φ), then by Fatou's lemma

$\lim_{n\to\infty} j(\tau_n v) = +\infty$.

If v∈D(Φ), it follows from (2.28) and (2.29) by applying Lebesgue's dominated convergence theorem that

$\lim_{n\to} j(\tau_n v) = j(v)$

From these convergence properties and from (2.25) , it follows, by taking the limit in (2.30), that

$$\left.\begin{aligned}&a(u^*,v-u^*) + j(v) - j(u^*) \ge L(v-u^*) \qquad \forall v \in V,\\&u^* \in V\end{aligned}\right\}\ldots\ldots\ldots(2.31)$$

Then u^* is a solution of (π) and from the uniqueness property we have u^*=u. This proves that $\lim_{n\to\infty} u_n = u$ weakly in V.

Let us show that $\phi(u), u\phi(u) \in L^1(\Omega)$.

Let v∈k_n. Then u_n+t(v-u_n)∈k_n ∀t∈]0,1[.

Replacing v by u_n+t(v-u_n) in (π_n) and dividing both sides of the inequality by t we obtain

$$a(u_n, v-u_n) + \int_\Omega \frac{\Phi(u_n + t(v-u_n)) - \Phi(u_n)}{t}dx \ge L(v-u_n) \qquad \forall v \in k_n \ldots(2.32)$$

since $\Phi \in C^1(\Re)$ and $\Phi' = \phi$ we have

$$\lim_{\substack{t\to 0\\ t\rangle 0}} \frac{\Phi(u_n + t(v-u_n)) - \Phi(u_n)}{t} = \phi(u_n)\cdot(v-u_n) \quad \text{a.e. } \ldots\ldots\ldots\ldots(2.33)$$

Moreover since Φ is convex, we also have ∀t∈]0,1],

$$\phi(u_n)(v-u_n) \le \frac{\Phi(u_n + t(v-u_n)) - \Phi(u_n)}{t} \le \Phi(v) - \Phi(u_n) \text{ a.e } \ldots\ldots\ldots(2.34)$$

From (2.33), (2.34) and using Lebesgue's dominated convergence Theorem in (2.32), we obtain

$$a(u_n, v-u_n) + \int_\Omega \phi(u_n)(v-u_n)dx \ge L(v-u_n) \qquad \forall\, v \in k_n \ldots\ldots\ldots\ldots(2.35)$$

Then taking v=0 in (2.35) we have

$$a(u_n,u_n)+\int_\Omega \phi(u_n)u_n dx \le L(u_n),$$

which implies

$$\int_\Omega \phi(u_n)u_n dx \le \frac{\|f\|^2}{\alpha} \quad \text{..................(2.36)}$$

But $\phi(v)v \ge 0 \quad \forall v \in V$. Hence $\phi(u_n)u_n$ is bounded in $L^1(\Omega)$. Moreover for some subsequence $(u_{n_k})_{n_k}$ of $(u_n)_n$ we have

$$\phi(u_{n_k})u_{n_k} \to \phi(u)u \quad \text{a.e. in } \Omega.$$

Then by Fatou's lemma we obtain $u\phi(u) \in L^1(\Omega)$ and this completes the proof of the proposition since $u\phi(u) \in L^1(\Omega)$ implies obviously that $\phi(u) \in L^1(\Omega)$.

In view of proving that (π) implies (P) we also need the following two lemma(Glowinski, 1976):-

Lemma 2.2(Mosco, 1973): The solution u of (π) is characterized by

$$\left.\begin{array}{l} a(u,v-u)+\int_\Omega \phi(u)(v-u)dx \ge L(v-u) \quad \forall v \in L^\infty(\Omega)\cap V, \\ u \in V, u\phi(u) \in L^1(\Omega) \end{array}\right\} \text{......(2.37)}$$

Lemma 2.3(Mosco, 1973): Let u be the solution of (π) . Then u is characterized by:

$$\left.\begin{array}{l} a(u,v)+\int_\Omega \phi(u)v dx = L(v) \quad \forall v \in L^\infty(\Omega)\cap V, \\ u \in V, \phi(u) \in L^1(\Omega). \end{array}\right\} \text{................(2.38)}$$

Corollary 2.2: If u is the solution of (π) then u is also a solution of (P).

Proof: we recall that $V^*=H^{-1}(\Omega)\subset D(\Omega)$ and that

$a(u,v)=\langle Au,v\rangle \quad \forall u,v\in V$ and $L(v)=\langle f,v\rangle$.

Let u be a solution of (π). Then u is characterized by (2.38) and since $D(\Omega)\subset V$ we obtain

$$\langle Au,v\rangle + \int_\Omega \phi(u)v dx = \langle f,v\rangle \quad \forall v \in D(\Omega) \text{.............(2.39)}$$

From (2.39) it follows that

$$Au+\phi(u)=f \quad \text{in } D(\Omega), \text{ ....(2.40)}$$

since Au and $f\in V^*$, we have $\phi(u) \in V^*$. Hence

$\phi(u) \in L^1(\Omega)\cap H^{-1}(\Omega)$ and from (2.40) we obtain that u is a solution of (P).

We observe that the unique solution of (π) is also a solution of (P). Now prove the reciprocal property, that is, every solution of (P) is a solution of (π) and hence (P) has a unique solution.

In order to prove this we shall use the following density lemma:-

Lemma 2.4: $D(\Omega)$ is dense in $V \cap L^{\infty}(\Omega)$, $V \cap L^{\infty}(\Omega)$ being equipped with the strong topology of V and the weake* topology of $L^{\infty}(\Omega)$.

Theorem 2.2: Under the a bove hypothesis on V , a(.,.), L(.), $\phi(.)$, Problems (π) and (P) are equivalent.

Proof: We have already proved that (π) implies (P). we need only to prove that (P) implies (π).

From the definition of (P) we have

$$\left.\begin{aligned} &a(u,v) + <\phi(u), v> = L(v) \qquad \forall v \in V, \\ &u \in V, \phi(u) \in H^{-1}(\Omega) \cap L^{1}(\Omega). \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.41)$$

It follows from (2.41) that

$$a(u,v) + \int_{\Omega} \phi(u) v dx = L(v) \qquad \forall v \in D(\Omega) \ldots\ldots\ldots\ldots\ldots\ldots (2.42)$$

If $v \in V \cap L^{\infty}(\Omega)$ we know from lemma 2.4 that there exists a sequence $(v_n)_n$, $v_n \in D(\Omega)$, such that

$\lim_{n \to \infty} v_n = v$ strongly in V, ……………………….(2.43)

$\lim_{n \to \infty} v_n = v$ in $L^{\infty}(\Omega)$ weak *. ……………………(2.44)

since $v_n \in D(\Omega)$ we have , from (2.42),

$$a(u,v_n) + \int_{\Omega} \phi(u) v_n dx = L(v_n) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.45)$$

It follows from (2.45) that $\lim_{n \to \infty} a(u,v_n) = a(u,v)$,

$\lim_{n \to \infty} L(v_n) = L(v)$ and since $\phi(u) \in L^{1}(\Omega)$, (2.44) implies that

$$\lim_{n \to \infty} \int_{\Omega} \phi(u) v_n dx = \int_{\Omega} \phi(u) v dx$$

Thus taking the limit in (2.45), we obtain

$$a(u,v) + \int_{\Omega} \phi(u) v dx = L(v) \qquad \forall v \in V \cap L^{\infty}(\Omega).$$

Therefore (P) implies (2.38) which implies in turn (π) . This completes the proof of the theorem.

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