5 Cauchy Sequences

Often the biggest problem with showing that a sequence converges using the techniques we have seen so far is that we must know ahead of time to what it converges. This is often a chicken and egg type problem because to prove a sequence converges, we must seemingly already know it converges. An escape from this dilemma is provided by Cauchy sequences.

Definition 5.1. A sequence \( a_n \) is a Cauchy sequence if for all \( \varepsilon > 0 \) there is an \( N \in \mathbb{N} \) such that \( n, m \geq N \) implies \( |a_n - a_m| < \varepsilon \).

Theorem 5.1. A sequence converges iff it is a Cauchy sequence.

Proof. \( (\Rightarrow) \) Suppose \( a_n \to L \) and \( \varepsilon > 0 \). There is an \( N \in \mathbb{N} \) such that \( g \geq N \) implies \( |a_n - L| < \varepsilon/2 \). If \( m, n \geq N \), then

\[
|a_m - a_n| = |a_m - L + L - a_n| \leq |a_m - L| + |L - a_n| < \varepsilon/2 \varepsilon/2 = \varepsilon.
\]

This shows \( a_n \) is a Cauchy sequence.

\( (\Leftarrow) \) Let \( a_n \) be a Cauchy sequence. First, we claim that \( a_n \) is bounded. To see this, let \( \varepsilon = 1 \) and choose \( N \in \mathbb{N} \) such that \( n, m \geq N \) implies \( |a_n - a_m| < 1 \). In this case, \( a_N - 1 < a_n < a_N + 1 \) for all \( n \geq N \), so \( \{a_n : n \geq N\} \) is a bounded set. The set \( \{a_n : n < N\} \), being finite, is also bounded. Since \( \{a_n : n \in \mathbb{N}\} \) is the union of these two bounded sets, it too must be bounded.

Because \( a_n \) is a bounded sequence, Corollary 4.6 implies it has a convergent subsequence \( b_n \to L \). Let \( \varepsilon > 0 \) and choose \( N \in \mathbb{N} \) so that \( n, m \geq N \) implies \( |a_n - a_m| < \varepsilon/2 \). There is a \( b_k = a_{m_k} \) such that \( m_k \geq N \) and \( |b_{m_k} - L| < \varepsilon/2 \). If \( n \geq N \), then

\[
|a_n - L| = |a_n - b_k + b_k - L| \leq |a_n - b_k| + |b_k - L| < |a_n - a_{m_k}| + \varepsilon/2 < \varepsilon/2 + \varepsilon/2 = \varepsilon.
\]

Therefore, \( a_n \to L \).

According to this theorem, we can prove that a sequence converges without ever knowing precisely to what it converges. An example of the usefulness of this idea is contained in the following definition and theorem.

Definition 5.2. A sequence \( a_n \) is contractive if there is a \( c \in (0, 1) \) such that \( |x_{k+1} - x_k| \leq c|x_k - x_{k-1}| \) for all \( k > 1 \).

Theorem 5.2. If a sequence is contractive, then it converges.

Proof. Let \( x_k \) be a contractive sequence with contraction constant \( c \in (0, 1) \).

We first claim that if \( n \in \mathbb{N} \), then

\[
|x_n - x_{n+1}| \leq c^{n-1}|x_1 - x_2|.
\]  

(2)

This is proved by induction. When \( n = 1 \), the statement is \( |x_1 - x_2| \leq c^0|x_1 - x_2| = |x_1 - x_2| \), which is trivially true. Suppose that \( |x_n - x_{n+1}| \leq c^{n-1}|x_1 - x_2| \)

...
for some \( n \in \mathbb{N} \). Then, from the definition of a contractive sequence and the inductive hypothesis,
\[
|x_{n+1} - x_{n+2}| \leq c|x_n - x_{n+1}| \leq c(c^{n-1}|x_1 - x_2|) = c^n|x_1 - x_2|.
\]
This shows the claim is true in the case \( n + 1 \). Therefore, by induction, the claim is true for all \( n \in \mathbb{N} \).

To show \( x_n \) is a Cauchy sequence, let \( \varepsilon > 0 \). Since \( c^n \to 0 \), we can choose \( N \in \mathbb{N} \) so that
\[
|c^n| \leq \frac{\varepsilon}{|x_1 - x_2|}.
\]
Let \( n > m \geq N \). Then
\[
|x_n - x_m| = |x_n - x_{n-1} + x_{n-1} - x_{n-2} + \cdots + x_{m+1} - x_m - x_m|
\leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m+1} - x_m|
\]
Now, use (2) on each of these terms.
\[
\leq c^{n-2}|x_1 - x_2| + c^{n-3}|x_1 - x_2| + \cdots + c^{m-1}|x_1 - x_2|
= |x_1 - x_2|(c^{n-2} + c^{n-3} + \cdots + c^{m-1})
\]
Apply the formula for a geometric sum.
\[
= |x_1 - x_2|c^{m-1}\frac{1 - c^{n-m}}{1 - c}
< |x_1 - x_2|c^{m-1}\frac{1}{1 - c}
\]
Use (3) to estimate the following.
\[
\leq |x_1 - x_2|c^{N-1}\frac{1}{1 - c}
< |x_1 - x_2|c^{N-1}\frac{\varepsilon}{x_1 - x_2}
= \varepsilon
\]
This shows \( x_n \) is a Cauchy sequence.

**Example 5.1.** Let \(-1 < r < 1 \) and define the sequence \( s_n = \sum_{k=0}^{n} r^k \). If \( r = 0 \), the convergent os \( s_n \) is trivial. So, suppose \( r \neq 0 \). In this case
\[
\frac{|s_n+1 - s_n|}{|s_n - s_{n-1}|} = \frac{r^{n+1}}{r^n} = |r| < 1.
\]
This shows \( s_n \) is contractive, and Theorem 5.2 implies it converges.

**Problem 17.** If \( x_n \) is a sequence and there is a \( c \geq 1 \) such that \( |x_{k+1} - x_k| > c|x_k - x_{k-1}| \) for all \( k > 1 \), then can \( x_n \) converge?