1. Transformation of Boundary-Value Problems of Laplace Equations to Integral Equations

This section starts with a brief discussion of integral equations. Potential theory is thus applied to transform boundary-value problems of Laplace equations of the first and the second kind into integral equations.

1.1 Integral Equations

An equation that contains an unknown function in the integrand is called an integral equation. If the unknown function occurs in a linear form, the equation is called a linear integral equation. For example, in the one-dimensional case, the equation

\[ \varphi(x) - \lambda \int_a^b K(x, \xi) \varphi(\xi) \, d\xi + f(x) \]  

is a linear integral equation. Here \( \lambda \) is a real-valued or complex-valued parameter. \( f(x) \) is a given function. \( K(x, \xi) \) is also a known function and is called the kernel of integral equations. Two independent variables \( x \) and \( \xi \) vary in the region \([a, b]\). \( \varphi(x) \) is an unknown function.

If \( f(x) \equiv 0 \), we have

\[ \varphi(x) = \lambda \int_a^b K(x, \xi) \varphi(\xi) \, d\xi \]  

(1.2)

which is called the associated homogeneous equation of Eq. (1.1). Equation (1.1) is called a non homogeneous equation, and \( f(x) \) is the non homogeneous term.

In \( n \)-dimensional space, integrals in the integral equations are \( n \)-multiple, the unknown function is a function of independent variables and the integration domain is a finite region. For example, counterparts of Eq. (1.1) and (1.2) are, in the three-dimensional case,

\[ \varphi(M) = \lambda \iiint_\Omega K(M, P) \varphi(P) \, d\Omega + f(M), \]  

(1.3)

\[ \varphi(M) = \lambda \iiint_\Omega K(M, P) \varphi(P) \, d\Omega, \]  

(1.4)
respectively. Here $\Omega$ stands for a three-dimensional finite region, and $M$ and $P$ are points in $\Omega$.

With known kernel $K(M,P)$ and non homogeneous term $g(M)$, the integral equation

$$\psi(M) = \lambda \iiint_{\Omega} K(M,P)\psi(P) \, d\Omega + g(M),$$

(1.5)

is called the transpose equation of Eq. (1.3).

When the unknown function occurs only in the integrands, the equation is called the Fredholm integral equation of the first kind. Examples are

$$\int_{a}^{b} K(x,\xi)\varphi(\xi) \, d\xi = f(x),$$

(1.6)

$$\iiint_{\Omega} K(M,P)\varphi(P) \, d\Omega = f(M).$$

An integral equation that involves the unknown function outside of the integrands is called a Fredholm integral equation of the second kind. Examples are Eqs. (1.1),(1.3) and (1.5).

If the kernel is symmetric such that $K(M,P) = K(P,M)$, the equation is called a symmetric equation. If the kernel satisfies

$$K(M,P) = \frac{H(M,P)}{r_{PM}^{\alpha}}, \quad 0 < \alpha < n,$$

it is called a weakly-singular kernel. The corresponding equation is called an integral equation with a weakly-singular kernel. Here $H(M,P)$ is a continuous function. $r_{PM}$ is the distance between $P$ and $M$ and $n$ is the dimension of the integration domain.

An equivalent definition of a weakly-singular kernel is

$$|K(M,P)| \leq \frac{C}{r_{PM}^{\alpha}}, \quad 0 < \alpha < n,$$

where $C$ is a constant. In three-dimensional space, the dimension of the integration domain is two instead of three if the integrals involved are surface integrals.
1.2 Transformation of Boundary-Value Problems into Integral Equations

For boundary-value problems of Laplace equations in a simple, regular domains we can obtain their solutions by using the Fourier method of expansion, separation of variables, the integral transformation or the method of Green functions. These methods do not work, however, for problems in domains that are not simple and regular. We normally use either of the following two methods for those problems:

1. Find an analytical expression that contains the undetermined function and satisfies the Laplace equation. The undetermined function is then determined by imposing the boundary conditions.
2. Find the function set of functions satisfying the CDS. The solution is then determined by constructing harmonic functions from the set. This method belongs to the direct category in mathematical equations of physics.

A typical example of the former is seeking solutions of boundary-value problems using the potential theory, i.e. by transforming boundary-value problems into integral equations. If $S$ is a surface and density functions $\varrho(P)$ and $\varpi(P)$ are continuous on $S$, the single-layer and the double-layer potentials are harmonic functions in both $\Omega$ (the region inside $S$) and $\Omega'$ (the region outside $S$). In particular, the single-layer potential is continuous in both $\Omega + S$ and $\Omega' + S$; its normal derivative also has the limit from both inside and outside of $S$. Therefore, we may use the potentials as solutions of boundary value problems of Laplace equations. Their density functions are undetermined functions and can be determined by imposing the boundary conditions.

Here we discuss this method for solving boundary-value problems of Laplace equations of the first and the second kind

$$\begin{cases}
\Delta u = 0, \\
u|_S = f(M),
\end{cases} \quad \text{or} \quad \begin{cases}
\Delta u = 0, \\
\frac{\partial u}{\partial n}|_S = f(M).
\end{cases}$$

where the $S$ is a surface and $f(M)$ is a continuous function.

First Internal Boundary-Value Problems (Dirichlet Internal problems)

Assume that the double-layer potential with the undetermined density function $\tau(P)$

$$u(M) = \iint_S \frac{\tau(P) \cos(PM, n)}{r_{PM}^2} \, dS$$  \hspace{1cm} (1.7a)
is a function that satisfies \( \Delta u = 0 \), \( M \in \Omega \) and \( u|_{S} = f(M) \), and is continuous in \( \Omega' + S \).

\[
u(M) = \iint_{S} \frac{\tau(P) \cos(PM, \mathbf{n})}{r_{PM}^2} \, dS - 2\pi \tau(M).
\]

To obtain a solution \( u \) that is continuous on \( \Omega' + S \) and satisfies \( u|_{S} = f(M) \), we should impose

\[
u(M) = f(M),
\]

so that the boundary-value problem is transformed into the problem of seeking the solution \( \tau(P) \) of

\[
f(M) = \iint_{S} \frac{\tau(P) \cos(PM, \mathbf{n})}{r_{PM}^2} \, dS - 2\pi \tau(M)
\]

or

\[
\tau(M) = \iint_{S} K(M, P) \tau(P) \, dS - \frac{1}{2\pi} f(M), \quad (1.7b)
\]

where

\[
K(M, P) = \frac{\cos(PM, \mathbf{n})}{2\pi r_{PM}^2}
\]

This is a Fredholm integral equation of the second kind regarding \( \tau(P) \). Once \( \tau(P) \) is available from Eq. (1.7b), the solution of the Dirichlet internal problems can thus readily be obtained from the double-layer potential (1.7a).

**First External Boundary-Value Problems (Dirichlet External Problems)**

Similar to the Dirichlet internal problems, we can use the double-layer potential as a function that satisfies \( \Delta u = 0 \), \( M \in \Omega' \) and \( u|_{S} = f(M) \), and is continuous in \( \Omega' + S \). The density function \( \tau(P) \) is determined such that

\[
\tau(M) = \iint_{S} -K(M, P) \tau(P) \, dS + \frac{1}{2\pi} f(M).
\]

Equation (1.8) is also a Fredholm integral equation of the second kind.
Second Internal Boundary-Value Problems (Neumann Internal Problems)
Assume that the single-layer potential with the undetermined density function \( \mu(P) \)
\[
 u(M) = \iiint_S \frac{\mu(P)}{r_{PM}} \, dS
\]
is a function that satisfies \( \Delta u = 0 \), \( M \in \Omega \) and \( \frac{\partial u}{\partial n} \), and is continuous on \( \Omega + S \). we have, for \( M \in S \)
\[
 \frac{\partial u}{\partial n} - \iiint_S \mu(P) \frac{\partial}{\partial n} \left( \frac{1}{r_{PM}} \right) \, dS + 2\pi \mu(M)
\]
\[
= - \iiint_S \frac{\mu(P) \cos(PM, n)}{r_{PM}^2} \, dS + 2\pi \mu(M). 
\]
Thus the Neumann internal problems are transformed into Fredholm integral equations of the second kind regarding
\[
\mu(M) = \iiint_S K(M, P) \mu(P) \, dS + \frac{1}{2\pi} f(M), \tag{1.9}
\]
where \( K(M, P) \) is the same as in the Dirichlet problems.

Second External Boundary-Value Problems (Neumann External Problems)
By following a similar approach as that used in obtaining Eq. (1.9), we can transform the Neumann external problems into Fredholm integral equations of the second kind regarding
\[
\mu(M) = \iiint_S -K(M, P) \mu(P) \, dS + \frac{1}{2\pi} f(M), \tag{1.10}
\]
Where
\[ f(M) = \frac{\partial u}{\partial n} \]
\( n \) is the outer normal for \( \Omega \) that is also the inner normal
Remark 1. In integral equations (1.7)–(1.10), the kernel $K(M,P)$ and its transpose $K(P,M)$ satisfy

$$K(M,P) = -K(P,M).$$

By using these relations, the four equations become

$$\tau(M) = \iint_S K(M,P)\tau(P)\,dS - \frac{1}{2\pi}f(M), \quad (1.7')$$

$$\tau(M) = -\iint_S K(M,P)\tau(P)\,dS + \frac{1}{2\pi}f(M), \quad (1.8')$$

$$\mu(M) = -\iint_S K(P,M)\mu(P)\,dS + \frac{1}{2\pi}f(M), \quad (1.9')$$

$$\mu(M) = \iint_S K(P,M)\mu(P)\,dS + \frac{1}{2\pi}f(M). \quad (1.10')$$

Therefore, Eqs. (1.7’) and (1.10’) are transpose equations of each other. Equations (1.8’) and (1.9’) are also transpose equations of each other.

### 1.4 Two-Dimensional Potential Equations

Dirichlet internal and external problems of Laplace equations in two-dimensional space read

$$\begin{cases}
\Delta u = 0, & M \in D, \\
u|_C = f(M)
\end{cases} \quad \text{and} \quad \begin{cases}
\Delta u = 0, & M \in D', \\
u|_C = f(M)
\end{cases} \quad (1.11)$$

respectively. Here $D$ and $D'$ are the plane domains inside of closed boundary curve $C$ and outside of $C$, respectively. The Neumann internal and external problems of Laplace equations can also be written out simply by replacing $u|_C = f(M)$ in Eq. (1.11) by

$$\frac{\partial u}{\partial n}|_C = f(M)$$

Following a similar approach for three-dimensional cases, we can transform these four problems into four integral equations regarding $\tau(M)$ or $\mu(M)$.
\[ \tau(M) = \int_C K(M, P) \tau(P) \, ds - \frac{1}{\pi} f(M), \]
\[ \tau(M) = -\int_C K(M, P) \tau(P) \, ds + \frac{1}{\pi} f(M), \quad \text{(1.12)} \]
\[ \mu(M) = -\int_C K(P, M) \mu(P) \, ds - \frac{1}{\pi} f(M), \]
\[ \mu(M) = \int_C K(P, M) \mu(P) \, ds + \frac{1}{\pi} f(M). \]

where, with \( r_{PM} \) as the distance between \( P \) and \( M \),

\[ K(M, P) = \frac{\cos(PM, n)}{\pi r_{PM}} \quad \text{(1.13)} \]

and thus,

\[ K(P, M) = -K(M, P) \]