7 MAGNETICALLY COUPLED CIRCUITS

7.1 Mutual Inductance

Consider two coils with self-inductances $L_1$ and $L_2$ that are in close proximity with each other (Fig. 7.1).

The magnetic flux $\phi_1$ emanating from coil 1 has two components: one component $\phi_{11}$ links only coil 1, and another component $\phi_{12}$ links both coils. Hence,

$$\phi_1 = \phi_{11} + \phi_{12}$$

Although the two coils are physically separated, they are said to be magnetically coupled. Since the entire flux $\phi_1$ links coil 1, the voltage induced in coil 1 is

$$v_1 = N_1 \frac{d\phi_1}{dt} = N_1 \frac{d\phi_{11}}{dt} \frac{di_1}{dt} = L_1 \frac{di_1}{dt}$$

Only flux $\phi_{12}$ links coil 2, so the voltage induced in coil 2 is

$$v_2 = N_2 \frac{d\phi_{12}}{dt} = N_2 \frac{d\phi_{12}}{dt} \frac{di_1}{dt} = M_{21} \frac{di_1}{dt}$$

where

$$M_{21} = N_2 \frac{d\phi_{12}}{di_1}$$

$M_{21}$ is known as the mutual inductance of coil 2 with respect to coil 1. To find the polarity of mutual voltage $M \frac{di}{dt}$, we apply the dot convention in circuit analysis. This is illustrated in Fig. 7.2. The dot convention is stated as follows:

1. If a current enters the dotted terminal of one coil, the reference polarity of the mutual voltage in the second coil is positive at the dotted terminal of the second coil.

2. If a current leaves the dotted terminal of one coil, the reference polarity of the mutual voltage in the second coil is negative at the dotted terminal of the second coil.
Figure 7.3 shows the dot convention for coupled coils in series. The total inductance is

(Series-aiding connection, Fig. 7.3a)

\[ L = L_1 + L_2 + 2M \]

(Series-opposing connection, Fig. 7.3b)

\[ L = L_1 + L_2 - 2M \]

The coupling coefficient \( k \) is a measure of the magnetic coupling between two coils and is given by

\[ k = \frac{M}{\sqrt{L_1 L_2}} \]

where \( 0 \leq k \leq 1 \) or equivalently \( 0 \leq M \leq \sqrt{L_1 L_2} \). The coupling coefficient is the fraction of the total flux emanating from one coil that links the other coil.

\[ k = \frac{\phi_{12}}{\phi_1} = \frac{\phi_{12}}{\phi_{11} + \phi_{12}} \]

If the entire flux produced by one coil links another coil, then \( k = 1 \) and we have 100 % coupling, or the coils are said to be perfectly coupled. For \( k < 0.5 \), coils are said to be loosely coupled; and for \( k > 0.5 \), they are said to be tightly coupled. We expect \( k \) to depend on the: (a) closeness of the two coils, (b) their core, (c) their orientation, (d) and their windings.

**Example 7.1:** Calculate the phasor currents \( I_1 \) and \( I_2 \) in the circuit of Fig. 7.4.

**Solution:**
For coil 1, KVL gives
\[-12 + (-j4 + j5)I_1 - j3I_2 = 0\]

or
\[ jI_1 - j3I_2 = 12 \]

For coil 2, KVL gives
\[-j3I_1 + (12 + j6)I_2 = 0\]

or
\[ I_1 = \frac{(12 + j6)I_2}{j3} = (2 - j4)I_2 \]

thus, \[ I_1 = 13.01 \angle -49.39^\circ \text{ A} \] and \[ I_2 = 2.91 \angle 14.04^\circ \text{ A} \]

### 7.2 Linear Transformers

The transformer (or air-core transformers) is said to be linear if the coils are wound on a magnetically linear material—a material for which the magnetic permeability is constant. Such materials include air, plastic, Bakelite, and wood. Applying KVL to the two meshes in Fig. 7.5 gives

\[ V = (R_1 + j\omega L_1)I_1 - j\omega M I_2 \tag{7.1} \]
\[ 0 = -j\omega M I_1 + (R_2 + j\omega L_2 + Z_L)I_2 \tag{7.2} \]

From Eq. (7.1) and (7.2), we get the input impedance as

\[ Z_{in} = \frac{V}{I_1} = R_1 + j\omega L_1 + \frac{\omega^2 M^2}{R_2 + j\omega L_2 + Z_L} \]

the reflected impedance \( Z_R \) is defined as

\[ Z_R = \frac{\omega^2 M^2}{R_2 + j\omega L_2 + Z_L} \]

We want to replace the linear transformer in Fig. 7.5 by an equivalent T or \( \Pi \) circuit, a circuit that would have no mutual inductance. Ignore the resistances of the coils and assume that the coils have a common ground as shown in Fig. 7.6.

The voltage-current relationships for the primary and secondary coils give the matrix equation

\[ \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} j\omega L_1 & j\omega M \\ j\omega M & j\omega L_2 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \tag{7.3} \]

By matrix inversion, this can be written as

\[ \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} \frac{L_2}{j\omega(L_1 L_2 - M^2)} & -\frac{M}{j\omega(L_1 L_2 - M^2)} \\ -\frac{M}{j\omega(L_1 L_2 - M^2)} & \frac{L_1}{j\omega(L_1 L_2 - M^2)} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \tag{7.4} \]
Our goal is to match Eqs. (7.3) and (7.4) with the corresponding equations for the T and \( \Pi \) networks.
For the T (or Y) network of Fig. 7.7, mesh analysis provides the terminal equations as

\[
\begin{bmatrix}
V_1 \\
V_2
\end{bmatrix} = \begin{bmatrix}
j\omega (L_a + L_c) & j\omega L_c \\
j\omega L_c & j\omega (L_b + L_c)
\end{bmatrix} \begin{bmatrix}
I_1 \\
I_2
\end{bmatrix}
\]  

(7.5)

If the circuits in Figs. 7.6 and 7.7 are equivalents, Eqs. (7.3) and (7.5) must be identical. This leads to

\[L_a = L_1 - M, \quad L_b = L_2 - M, \quad L_c = M\]

For the \( \Pi \) (or \( \Delta \)) network in Fig. 7.8, nodal analysis gives the terminal equations as

\[
\begin{bmatrix}
I_1 \\
I_2
\end{bmatrix} = \begin{bmatrix}
\frac{1}{j\omega L_A} + \frac{1}{j\omega L_C} & -\frac{1}{j\omega L_C} \\
-\frac{1}{j\omega L_C} & \frac{1}{j\omega L_B} + \frac{1}{j\omega L_C}
\end{bmatrix} \begin{bmatrix}
V_1 \\
V_2
\end{bmatrix}
\]  

(7.6)

Equating terms in admittance matrices of Eqs. (7.4) and (7.6), we obtain

\[L_A = \frac{L_1 L_2 - M^2}{L_2 - M}, \quad L_B = \frac{L_1 L_2 - M^2}{L_1 - M}, \quad L_C = \frac{L_1 L_2 - M^2}{M}\]

Note that in Figs. 7.7 and 7.8, the inductors are not magnetically coupled.

**Example 7.2:** Solve for \( I_1, I_2 \), and \( V_o \) in Fig. 7.9 using the T-equivalent circuit for the linear transformer.

**Solution:** First, due to the current reference directions and voltage polarities, we need to replace \( (M) \) by \( (-M) \). Therefore;

\[L_a = L_1 - (-M) = 8 + 1 = 9 \text{ H}\]
\[L_b = L_2 - (-M) = 5 + 1 = 6 \text{ H}\]
\[L_c = -M = -1 \text{ H}\]