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INTRODUCTION

It has been shown that electromagnetic waves in matter travel at a speed that is reduced, and can be expressed in terms of the refractive index \( n \):

\[
v = \frac{c}{n}
\]

where Maxwell’s equations were used to predict that the refractive index \( n \) is given by:

\[
n = \sqrt{nEM}
\]

For non-ferromagnetic materials the relative permeability is very close to unity and then \( n = \sqrt{\varepsilon} \) where \( \varepsilon \) is the dielectric constant. The plane EM wave in matter has the relation between the perpendicular coupled electromagnetic fields \( E = vB \) and travels in the \( E \times B \) direction.

In the absence of any obstructions or media effects, electromagnetic radiation travels in straight lines. The fact that light travels in straight lines, and casts a sharp shadow, led Newton to believe that light involved particle motion not wave motion. Solutions of the wave equation also predict electromagnetic waves traveling in a straight line. For example, we saw that a plane wave of the form \( \mathbf{E} = E_0 \hat{\mathbf{i}} \sin(kz - \omega t) \) travels in the \( z \) direction. However, last lecture it was shown that waves behave differently when obstructions are encountered. The difference between particle and wave motion is that, for obstructions comparable in size to the wavelength, waves are diffracted outwards at the edges. The angle of the first minimum for diffraction at a slit of width \( d \) is given by \( \sin \theta = \frac{1}{2} \). Diffraction effects become negligible when \( d \gg \lambda \), which is the realm of geometric optics. For many applications of wave propagation, such as optics, diffraction effects can be ignored, to first order, allowing use of geometrical optics, the topic of today’s lecture.

Although the propagation of waves can be predicted exactly using mathematical solutions of the wave equation, with known boundary conditions, you may find it helpful to consider the Huygens construct of secondary wavelets to explain wave motion. The Huygens construct assumes that each point on a primary wavefront serves as the source of spherical secondary wavelets that advance with the speed and frequency of the primary wavefront. Figure 1 illustrates the Huygens construct for the propagation of plane and spherical waves. The secondary wavefront is the locus of the secondary wavelets. Note that the wavelets do spread outwards into the shadow at the edge of the plane wave. Calculation of the secondary wavefront requires including the relative phases and amplitudes of the wavelets involved. The Huygens construct is useful in understanding reflection, refraction and diffraction. The trajectory taken by an electromagnetic wave also is predicted by Fermat’s principle which states that the wave-front follows a path that minimizes the transit time.

REFLECTION AND REFRACTION AT BOUNDARIES

The reflection and transmission of electromagnetic waves incident normally at the boundary between two media has been discussed and the reflectivity and transmission calculated. When waves are incident upon a boundary at non-normal incidence, the reflected and transmitted waves are emitted at definite non-normal angles, and the reflectivity and transmission differ from the case of normal incidence discussed earlier. The angles of incidence, reflection and refraction are most conveniently measured with respect to the normal to the boundary surface. The plane of incidence is the surface defined by the incident ray and the normal to the surface.
Wave with velocity $v_1$ in medium 1 incident upon a boundary with medium 2 in which the wave has velocity $v_2$. The angle of incidence to the normal to the surface is $\theta_I$, the angle of reflection to the normal is $\theta_R$, and the angle of refraction relative to the normal is $\theta_T$.

### Reflection

Reflected and incident waves are in the same material and therefore have the same velocity. Therefore, using the Huygens construct shown in figure 2, it can be seen that the reflected wave is in the plane of incidence and the angle of reflection equals the angle of incidence.

\[ \theta_R = \theta_I \]

### Refraction

The transmitted wave travels in a different medium, with different speed compared to the incident wave. Using the Huygens construct, one can see that in a given time $\Delta t$, the incident and transmitted waves travel the distances $v_I \Delta t$ and $v_T \Delta t$ respectively. Since the two wavefronts have a common hypotenuse, then we have the relation that gives the angle of refraction of the transmitted wave:

\[ \frac{\sin \theta_T}{\sin \theta_I} = \frac{v_T h}{v_I h} \]

that is:

\[ \frac{\sin \theta_T}{v_T} = \frac{\sin \theta_I}{v_I} \]

These two relations for the angles of reflection and refraction are true for any type of travelling wave. For electromagnetic waves, the velocity can be expressed in terms of the refractive index, that is $v = \frac{c}{n}$. Using this, leads to Snell’s Law that gives the refracted wave is in the plane of incidence at an angle of refraction given by:

\[ n_I \sin \theta_I = n_T \sin \theta_T \quad \text{(Snell’s Law)} \]

As an example, consider light in air incident upon glass of refractive index $n = 1.5$ as shown in figure 3. For normal incidence $\theta_I = \theta_R = \theta_T = 0$. For an incident angle $\theta_I = 45^\circ$, the angle of reflection is $45^\circ$ and the angle of refraction is given by:

\[ \sin \theta_T = \frac{1}{1.5} \sin 45^\circ \]

which gives $\theta_T = 28.13^\circ$. At the maximum angle of incidence $\theta_I = 90^\circ$, Snell’s law gives $\theta_T = 41.81^\circ$.

### Critical angle

When electromagnetic radiation strikes a boundary going from higher to lower refractive index, the transmitted wave is bent away from the normal at a larger angle than the angle of incidence as shown in figure 4. For incident angles at the critical angle, $\theta_c$, the angle of refraction is $90^\circ$. That is,

\[ \sin \theta_c = \frac{n_2}{n_1} \quad \text{(Critical angle)} \]

For angles of incidence $\theta_I > \theta_c$ there is no solution to Snell’s law since there is no angle for which the sine exceeds unity. For angles of incidence greater than the critical angle, the incident wave is totally internally reflected, there is no transmitted wave. Thus the surface behaves as a perfect mirror. Underwater swimmers are familiar with this phenomenon, they can see upwards through only an angular region of the water surface within $\theta = 48.8^\circ$ of the normal to the surface. For glass, $n = 1.5$, the critical angle is
\[ \theta_e = \sin^{-1}\left(\frac{1}{n}\right) = 41.8^\circ. \] 
Figure 5 shows that light can be totally internally reflected twice to exactly reverse the direction of travel using a right-angle prism. Four such prisms are used in binoculars to correct for inversion of the image by the lens. An important technical application of total internal reflection is the optical fiber illustrated in figure 6. This has applications in medicine as well as for long-distance transmission of signals.

**Polarization**

Since electromagnetic waves are transverse waves it can be shown that when the angle of reflection plus angle of refraction sum to 90°, the reflected wave is plane polarized as illustrated in figure 7. The reflected wave is fully plane polarized at the Brewster angle given by:

\[ \tan \theta_{\text{Brewster}} = \frac{n_2}{n_1}, \]

Polaroid sunglasses can be used to attenuate the polarized reflected light from a surface such as a window.

## REFLECTION AT SINGLE SURFACE

Geometrical optics for reflective systems, that is when \( \lambda << d \), is dominated by the statement that the reflected wave is in the plane of incidence and the angle of reflection equals the angle of incidence. Let us consider simple reflective systems.

**Plane mirror**

As shown in figure 8, radiation reflected from the mirror appears to emanate from a virtual image that is the same distance behind the mirror as is the object in front of the mirror. An important feature of the mirror image is that depth inversion results in a right-handed system, \( \mathbf{i} \times \mathbf{j} = \mathbf{k} \) having a left-handed mirror image where \( \mathbf{i} \times \mathbf{j} = -\mathbf{k} \). This is illustrated in figure 9. The behaviour of physics under transformation of a right-handed system into a left-handed mirror-image system is an important fundamental symmetry of nature.

**Concave mirrors**

Consider the concave spherical surface of radius \( r \) shown in figure 10. Let the object be a distance \( O \) and image a distance \( I \) from the mirror. From elementary
geometry the exterior angle $\beta$ of the triangle $PAC$ is:

$$\beta = \alpha + \theta$$

Similarly:

$$\gamma = \alpha + 2\theta$$

Eliminating $\theta$ from these equations gives

$$2\beta = \alpha + \gamma$$

Using the small angle approximation; $\alpha \approx \frac{1}{f} \beta \approx \frac{1}{r}$

$\gamma \approx \frac{1}{r}$, gives:

$$\frac{1}{O} + \frac{1}{I} = \frac{2}{r}$$

Note that if $O \to \infty$ then the image distance is called the focal length $f$ where:

$$f = \frac{1}{2r}$$

Written in terms of the focal length, the paraxial (small angle) approximation gives:

$$\frac{1}{O} + \frac{1}{I} = \frac{1}{f}$$

(Mirror equation)

Note that all paraxial rays from $O$ converge on $I$ and vise versa. For an object of height $o$, the image is inverted and has height $i = -\frac{o}{f}$ as shown in figure 11. The transverse magnification defined as the ratio of the image to object sizes is:

$$M = -\frac{I}{O}$$

(Magnification)

Sign convention

The sign convention used for reflective systems is important. The sign convention assumed is: $I, O, r$ and $f$ are positive if they are in front of the mirror, and negative behind the mirror.

As an example, consider a concave mirror with $r = 10cm$, $f = 5cm$, if $O = 25cm$ then:

$$\frac{1}{f} = \frac{1}{O} = \frac{1}{5} - \frac{1}{25} = \frac{4}{25}$$

Thus $I = 6.25cm$ and the magnification $M$ is:

$$M = \frac{I}{O} = -\frac{25}{4x25} = -0.25$$

However, if $O = 4cm$, that is $O < f$, then

$$\frac{1}{f} = \frac{1}{O} = \frac{1}{5} - \frac{1}{4} = -\frac{1}{20}$$

That is, $I = -20cm$, it is a virtual image behind the mirror. The magnification $M = +\frac{20}{-20} = +5$. That is, the image is erect, not inverted as illustrated in figure 12.

Ray tracing

It is recommended that simple geometry contructions be used to trace rays through an optical system as illustrated in figure 13 since it gives a useful visual representation of the focussing properties of any optical system. The following are the suggested way to trace rays through a reflective system.

1) A ray emitted parallel to the optical axis goes through the corresponding focal point as shown by the red ray-1 trajectory in figure 13.

2) A ray that passes through the nearby focal point becomes parallel to the optical axis as shown by the yellow ray-2 trajectory.

3) A ray that passes through the radius of curvature of the mirror is reflected back through the center of the mirror as shown by the purple ray-3.
4) A ray that hits the mirror on the optical axis is reflected back at the same angle to the optical axis as shown by the blue ray-4.

Figure 13 illustrates the focusing properties of a concave mirror for the cases leading to either a real or a virtual image.

Convex mirrors

Figure 14 shows the geometry for a convex mirror. Here the focal length and radius of curvature are negative since they are behind the mirror surface. Again assuming the paraxial approximation gives that

\[ \frac{1}{O} + \frac{1}{f} = \frac{1}{r} \]

where \( O \) is positive, while \( I \) and \( f \) are negative. The magnification now is a positive number since \( I \) and \( O \) have opposite signs, thus the image is erect, not inverted.

Applications

The big advantage of reflecting mirrors is that there is no dispersion and mirrors are cheaper to build than lenses. Concave mirrors are used extensively for radio and optical telescopes. Convex mirrors are used as rear-view mirrors on cars because they have a wide angle and erect image. Whispering galleries are examples of use of concave mirrors to focus sound waves.

REFRACTION AT A SINGLE BOUNDARY

Geometric optics for refractive systems is dominated by Snell’s Law, the refracted wave is in the plane of incidence at an angle of refraction given by:

\[ n_I \sin \theta_I = n_T \sin \theta_T \]  

(Snell’s Law)

Convex refracting surface

Consider refraction at a spherical surface separating two media of refractive index \( n_1 \) and \( n_2 \) as shown in figure 15. Refraction depends on Snell’s law:

\[ n_I \sin \theta_I = n_T \sin \theta_T \]

Assuming the small-angle approximation means that \( \sin \theta \approx \theta \). In the small-angle approximation Snell’s law simplifies to:

\[ n_1 \theta_1 = n_2 \theta_2 \]  

(Small-angle approximation)

Now using elementary geometry one sees that:

\[ \theta_1 = \alpha + \beta \]

and

\[ \beta = \theta_2 + \gamma \]

Substitution for \( \theta_2 \) using Snell’s law gives:

\[ \beta = \frac{n_1}{n_2} \theta_1 + \gamma \]

Eliminating \( \theta_1 \) gives:

\[ n_1 \alpha + n_2 \gamma = (n_2 - n_1) \beta \]

The paraxial ray assumption allows the approximation that:

\[ \alpha \approx \frac{r}{O}, \; \beta \approx \frac{r}{r}, \; \gamma \approx \frac{r}{T} \]

Thus this gives the paraxial refraction equation:

\[ \frac{n_1}{O} + \frac{n_2}{T} = \frac{(n_2 - n_1)}{r} \]
Sign convention

The sign convention assumed for refraction is:

O: positive for a real object on the incident side of the surface, negative for a virtual object on the transmission side of the surface

I: positive for a real image on the transmission side of the surface, negative for a virtual image on the incident side of the surface.

r,f: positive if the center of curvature is on the transmission side, negative if the center of curvature is on the incident side.

Note that the sign convention for reflection has the real image on the incident side as opposed to the lens where the real image is on the transmission side.

Plane surface

An interesting case of refraction at a single surface is for a plane surface for which $r = \infty$, shown in figure 16. Using the above lens equation gives:

\[
\frac{n_1}{O} + \frac{n_2}{I} = 0
\]

Thus the image distance is given by:

\[
I = -\frac{n_2}{n_1}O
\]

For example, light coming from an object a distance $O$ below the surface of water creates a virtual image that appears to be a distance $\frac{1}{1.33}O$ below the surface. That is, the object appears closer to the surface.

Using the paraxial approximation then the object size $o$, and image size, $i$, then using figure 17, $\theta_1 \approx \frac{o}{O}$ and $\theta_2 \approx \frac{i}{I}$, plus Snell's law $n_1 \theta_1 = n_2 \theta_2$, gives a magnification:

\[
M = \frac{i}{o} = -\frac{n_1 I}{n_2 O}
\]

Note that the magnification is unity for the case of a plane surface.

DISPERSION

The variation of refractive index with frequency was discussed previously and is illustrated in figure 18. The angle of refraction depends on the refractive index as given by Snell's law. Thus the angle of refraction of blue light is greater than that of red light leading to a dispersion of the component colours or wavelengths of light as illustrated in figure 19.

The rainbow is a nice example of total internal reflection plus dispersion as illustrated in figure 20. Total internal reflection leads to a maximum reflection at about 180-42° relative to the angle of incidence of sunlight on a water droplet. Dispersion occurs due to refraction for the light beam entering and leaving the droplet. Thus one observes the rainbow at an angle 180-42° relative to the sunlight with the red ring on the outside of the rainbow and violet on the inside of
the rainbow. If you look carefully you will observe a secondary rainbow due to two internal reflections inside the raindrop. This secondary rainbow is outside the primary rainbow and, due to the second total internal reflection, has the opposite sequence of colors.

**SUMMARY**

The image formed by a spherical mirror or lens system has been derived leading to the lens formula.

\[
\frac{1}{O} + \frac{1}{I} = \frac{1}{f}
\]

Care must be taken applying this simple formula to ensure that the sign convention is obeyed. The image properties of complicated lens or mirror systems can be obtained by cascading application of the thin lens formula to each lens or reflecting surface in series. Next lecture will combine geometric optics with diffraction effects for a discussion of optical instruments.

**Reading assignment:** Giancoli, Chapter 33.