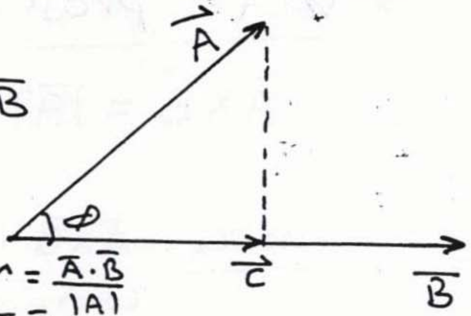
* projection

$\bar{C}$ = vector projection of $\vec{A}$ onto $\bar{B}$



$$\text{proj}_{\bar{B}}\,\vec{A} = \frac{\bar{A}\cdot\bar{B}}{\bar{B}\cdot\bar{B}}\cdot\bar{B}$$

$$\text{Proj}_{\bar{A}}\,\bar{B} = \frac{\bar{A}\cdot\bar{B}}{\bar{A}\cdot\bar{A}}\cdot\bar{A}$$

but scalar component:

$\bar{B}$ comp. in $\bar{A}$ direction $= \dfrac{\bar{A}\cdot\bar{B}}{|A|}$

$\bar{A}$ " " $\bar{B}$ " $= \dfrac{\bar{A}\cdot\bar{B}}{|B|}$

EX/ If $\vec{A} = i-2j-2k$, $\vec{B} = 6i+3j+2k$ Find

ⓐ $\bar{A}\cdot\bar{B}$ ⓑ angle between Them ⓒ component of $\bar{B}$ in $\bar{A}$ direction ⓓ component of $\bar{A}$ in $\bar{B}$ direction.

sol/

ⓐ $\bar{A}\cdot\bar{B} = 1*6+(-2)*3+(-2)*2 = -4$

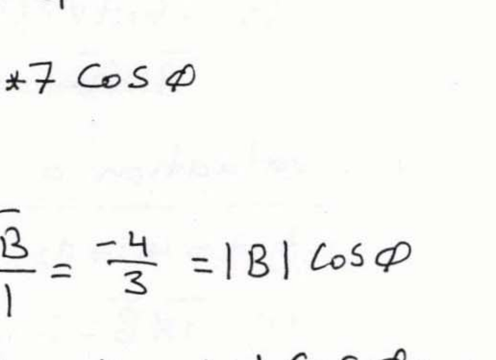
ⓑ $\bar{A}\cdot\bar{B} = |\bar{A}||\bar{B}|\cos\phi \Rightarrow -4 = 3*7\cos\phi$

$$\therefore \phi = \cos^{-1}\frac{-4}{21} = +101^{\circ}$$

ⓒ $\bar{B}$ comp. in $\bar{A}$ direction $= \dfrac{\bar{A}\cdot\bar{B}}{|A|} = \dfrac{-4}{3} = |B|\cos\phi$

ⓓ $\bar{A}$ " " $\bar{B}$ " $= \dfrac{\bar{A}\cdot\bar{B}}{|B|} = \dfrac{-4}{7} = |A|\cos\phi$.

EX/ prove That $d = \dfrac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}$ is The perpendicular distance from The point $P_0(x_0, y_0)$ to The $ax+by+c=0$.



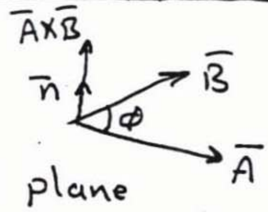
sol/ $\overrightarrow{AP_0} = x_0 i + (y_0 + \frac{c}{b}) j$

$$d = \text{proj}_{N}\,\overrightarrow{AP_0} = \frac{|\overrightarrow{AP_0}\cdot\vec{N}|}{|N|}$$

$$= \frac{|a(x_0)+b(y_0+\frac{c}{b})|}{\sqrt{a^2+b^2}} = \frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}$$

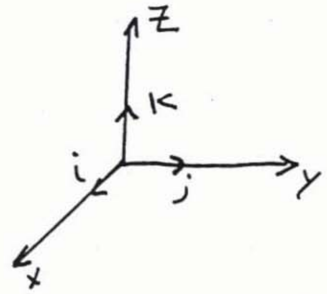
* Vector product of two vectors (Cross product)

$$\bar{A} \times \bar{B} = |\bar{A}||\bar{B}| \sin \phi \cdot \bar{n}$$



Hence : $\bar{A}\times\bar{B}$ is a vector normal to the plane containing $\bar{A}$ & $\bar{B}$ & has the magnitude of $|\bar{A}||\bar{B}|\sin\phi$

Note ① $\bar{A}\times\bar{B} = -\bar{B}\times\bar{A}$



② $i\times j = k$ $\quad j\times i = -k$ $\quad$ & $\quad i\times i = 0$
$\quad i\times k = -j$ $\quad k\times i = j$ $\quad j\times j = 0$
$\quad j\times k = i$ $\quad k\times j = -i$ $\quad k\times k = 0$

③ $(c_1\bar{A})\times(c_2\bar{B}) = (c_1c_2)\bar{A}\times\bar{B}$ --- associative Law.

$\bar{A}\times(\bar{B}+\bar{C}) = \bar{A}\times\bar{B} + \bar{A}\times\bar{C}$ --- distributive Law.

* Evaluation of $\bar{A}\times\bar{B}$ interms of component of $\bar{A}$ & $\bar{B}$:

let $\vec{A} = a_1 i + a_2 j + a_3 k$ & $\vec{B} = b_1 i + b_2 j + b_3 k$

Then: $\bar{A}\times\bar{B} = (a_1 i + a_2 j + a_3 k)\times(b_1 i + b_2 j + b_3 k)$

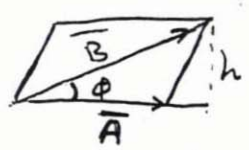
$$= a_1 b_2 k + (-a_1 b_3 j) + (-a_2 b_1) k + a_2 b_3 i + a_3 b_1 j + (-a_3 b_2) i$$

$$= (a_2 b_3 - a_3 b_2) i + (a_3 b_1 - a_1 b_3) j + (a_1 b_2 - a_2 b_1) k$$

$$\therefore \bar{A}\times\bar{B} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

* Area of parallelogram $= |\bar{A}\times\bar{B}| = |\bar{A}||\bar{B}|\sin\phi$

since $h = |\bar{B}|\sin\phi$



* Area of triangle $= \dfrac{|\bar{A}\times\bar{B}|}{2}$

EX/ Find The area of triangle ABC given That:

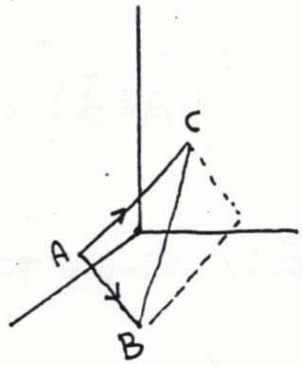
$A(1,-1,0)$, $B(2,1,-1)$, $C(-1,1,2)$

sol/ area of ABC $= \frac{1}{2} |\overline{AB} \times \overline{AC}|$

$\overline{AB} = i+2j-K$ & $\overline{AC} = -2i+2j+2K$

$\overline{AB} \times \overline{AC} = 6i + 6K$

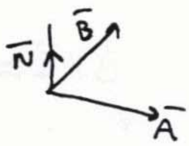
$\therefore$ area $= 3\sqrt{2}$ unit area



EX/ Find The normal vector to

$\overline{A} = 2i-3j+4K$, $\overline{B} = i-2j+5K$

$$\overline{A} \times \overline{B} = \overline{N} = \begin{vmatrix} i & j & K \\ 2 & -3 & 4 \\ 1 & -2 & 5 \end{vmatrix} = -7i-6j-K$$



HW/ prove $\vec{A} \perp \overline{N}$

<u>Note</u> $(\overline{A} \times \overline{B}) \cdot \overline{A} = 0 \Rightarrow (\overline{A} \times \overline{B}) \perp \overline{A}$

$(\overline{A} \times \overline{B}) \cdot \overline{B} = 0 \Rightarrow (\overline{A} \times \overline{B}) \perp \overline{B}$

$(\overline{A} \times \overline{B}) \cdot \overline{C} \neq 0$ C not parallel with $\overline{A}$ or $\overline{B}$

If $\overline{A} \parallel \overline{B}$ Then $(\overline{A} \times \overline{C}) \cdot \overline{B} = 0$

HW/

① Find $\overline{A} \cdot \overline{B}$, $\overline{A} \times \overline{B}$ For $\overline{A} = i+3j+2K$, $\overline{B} = K$

② let $\vec{A} = i+2j-K$, $\overline{B} = -i+j+K$, $\overline{C} = i+K$, $\overline{D} = i+2j-K$

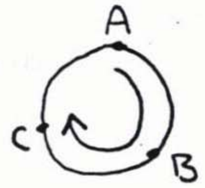
which vectors are: ⓐ perpendicular ⓑ parallel

③ If $\overline{A} = 2i-j$ and $\overline{B} = i+3j-2K$ Find $\overline{A} \times \overline{B}$, $(\overline{A} \times \overline{B}) \cdot \overline{A}$, and $(\overline{A} \times \overline{B}) \cdot \overline{B}$.

Product of Three-vectors. (1) Triple scalar Product

$(\bar{A} \times \bar{B}) \cdot \bar{C}$ called triple scalar product

Note $(\bar{A} \times \bar{B}) \cdot \bar{C} = (\bar{B} \times \bar{C}) \cdot \bar{A} = (\bar{C} \times \bar{A}) \cdot \bar{B}$



$$(\bar{A} \times \bar{B}) \cdot \bar{C} = \bar{N} \cdot \bar{C} = |\bar{N}||\bar{C}| \cos\phi = \text{volume}$$

EX/ show That $\bar{A} \cdot (\bar{B} \times \bar{C}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

sol/

$\bar{A} = a_1 i + a_2 j + a_3 k$, $\bar{B} = b_1 i + b_2 j + b_3 k$, $\bar{C} = c_1 i + c_2 j + c_3 k$

$$\bar{B} \times \bar{C} = \begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} i - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} j + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} k.$$

$$\bar{A} \cdot (\bar{B} \times \bar{C}) = \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} a_1 - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} a_2 + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} a_3$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(2) Triple vector product

$(\bar{A} \times \bar{B}) \times \bar{C}$ or $(\bar{A} \times \bar{C}) \times \bar{B}$ are Called triple vector product

where $(\bar{A} \times \bar{B}) \times \bar{C} = (\bar{A} \cdot \bar{C})\bar{B} - (\bar{B} \cdot \bar{C})\bar{A}$

& $(\bar{B} \times \bar{C}) \times \bar{A} = (\bar{B} \cdot \bar{A})\bar{C} - (\bar{C} \cdot \bar{A})\bar{B}$

EX/ show That $(\bar{A} \times \bar{B}) \times \bar{C} = (\bar{A} \cdot \bar{C})\bar{B} - (\bar{B} \cdot \bar{C})\bar{A}$ If

$\bar{A} = i - j + 2k$, $\bar{B} = 2i + j + k$, $\bar{C} = i + 2j - k$

sol/ $\bar{A} \cdot \bar{C} = -3$, $\bar{B} \cdot \bar{C} = 3$

R.H.S $(\bar{A} \cdot \bar{C})\bar{B} - (\bar{B} \cdot \bar{C})\bar{A} = -3(2i + j + k) - 3(i - j + 2k) = 9i - 9k$

L.H.S $\bar{A} \times \bar{B} = \begin{vmatrix} i & j & k \\ 1 & -1 & 2 \\ 2 & 1 & 1 \end{vmatrix} = -3i + 3j + 3k$

$$(\bar{A} \times \bar{B}) \times \bar{C} = \begin{vmatrix} i & j & k \\ -3 & 3 & 3 \\ 1 & 2 & -1 \end{vmatrix} = -9i - 9k$$

Solved problems

① Find a vector $\overrightarrow{N}$ perpendicular to The plane determined by The points $A(1,-1,2)$, $B(2,0,-1)$ & $C(0,2,1)$.

sol/

$$\overline{AB} = i+j-3K \quad \& \quad \overrightarrow{AC} = -i+3j-K$$

$$\therefore \overline{N} = \overline{AB} \times \overline{AC} = \begin{vmatrix} i & j & K \\ 1 & 1 & -3 \\ -1 & 3 & -1 \end{vmatrix} = 8i+4j+4K$$

② Find The vector projection of $\overrightarrow{B} = 6i+3j+2K$ onto $\overrightarrow{A} = i-2j-2K$ & The scalar component of $\overrightarrow{B}$ in The direction of $\overrightarrow{A}$.

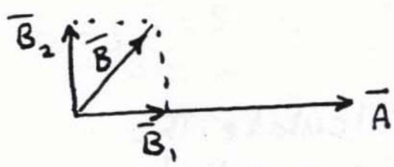
sol/ • $$\operatorname{proj}_A B = \frac{\overline{A}\cdot\overline{B}}{\overline{A}\cdot\overline{A}}\overline{A} = \frac{6-6-4}{1+4+4}(i-2j-2K) = \frac{-4}{9}i+\frac{8}{9}j+\frac{8}{9}K$$

• scalar component of $\overrightarrow{B}$ in The direction of $\overrightarrow{A} = |\overrightarrow{B}|\cos\theta = \overrightarrow{B}\cdot\frac{\overline{A}}{|\overline{A}|}$

$$= (6i+3j+2K)\cdot\left(\tfrac{1}{3}i-\tfrac{2}{3}j-\tfrac{2}{3}K\right) = 2-2-\frac{4}{3} = \frac{-4}{3}$$

③ Express $\overline{B} = 2i+j-3K$ as The sum of a vector parallel to $\overrightarrow{A} = 3i-j$ & a vector orThogonal to $\overline{A}$.

sol/ $\overline{A}\cdot\overline{B} = 6-1 = 5$ & $\overline{A}\cdot\overline{A} = 9+1 = 10$



$$\overline{B} = \overline{B}_1 + \overline{B}_2 = \frac{\overline{A}\cdot\overline{B}}{\overline{A}\cdot\overline{A}}\overline{A} + \left(\overline{B} - \frac{\overline{A}\cdot\overline{B}}{\overline{A}\cdot\overline{A}}\overline{A}\right)$$

$$= \frac{5}{10}(3i-j) + \left[2i+j-3K - \frac{5}{10}(3i-j)\right]$$

$$= \left(\frac{3}{2}i - \frac{1}{2}j\right) + \left(\frac{1}{2}i + \frac{3}{2}j - 3K\right)$$

<u>check</u> That is orThogonal

$$\left(\frac{1}{2}i+\frac{3}{2}j-3K\right)\cdot(3i-j) = \frac{3}{2}-\frac{3}{2} = 0$$