* Eigenvalues and Eigenvectors

In many applications of matrices involving coupled oscillations and vibrations, equations of the form

$A \cdot X = \lambda X$ where A is a square matrix, λ is a number.

At $X \neq 0$ the values of λ are called eigenvalues, and the solutions of $A \cdot X = \lambda X$ are called the eigenvectors.

$A \cdot X = \lambda X$ <u>or</u> $A \cdot X - \lambda X = 0 \Rightarrow (A - \lambda I) X = 0$

where $X \neq 0$ $\therefore |A - \lambda I| = 0$

$|A - \lambda I|$ called characteristic determinant of A

$|A - \lambda I| = 0$ = = equation.

Each eigenvalue (λ) has corresponding to it a solution of X called an eigenvector.

EX/ Find eigenvalues of $A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$

Sol/ $|A - \lambda I| = \begin{vmatrix} (4-\lambda) & -1 \\ 2 & (1-\lambda) \end{vmatrix} = 0$

$\therefore (4-\lambda)(1-\lambda) + 2 = 0 \Rightarrow \lambda^2 - 5\lambda + 6 = 0 \Rightarrow (\lambda-2)(\lambda-3) = 0$

$\therefore \lambda_1 = 2, \quad \lambda_2 = 3$

HW/ Find the eigenvalues of ① $A = \begin{bmatrix} 2 & 3 & -2 \\ 1 & 4 & -2 \\ 2 & 10 & -5 \end{bmatrix}$

② $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ -2 & 1 & -1 \end{bmatrix}$

Ans/ ① $\lambda_1 = -1, \quad \lambda_2 = 2, \quad \lambda_3 = 1$

② $\lambda_1 = -1, \quad \lambda_2 = 1, \quad \lambda_3 = 2$

EX/ Find The eigenvector of $A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$

Sol/ $\begin{vmatrix} (4-\lambda) & 1 \\ 3 & (2-\lambda) \end{vmatrix} = 0 \Rightarrow (4-\lambda)(2-\lambda) - 3 = 0$

$$\lambda^2 - 6\lambda + 5 = 0$$

$$(\lambda-1)(\lambda-5) = 0 \quad \therefore \lambda = 1, 5$$

For $\lambda = 1 \Rightarrow A \cdot X = \lambda X$

$$\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 1 \cdot \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

$$\left. \begin{aligned} 4X_1 + X_2 &= X_1 \\ 3X_1 + 2X_2 &= X_2 \end{aligned} \right\} \text{any equation gives } X_2 = -3X_1$$

$\therefore X_1 = \begin{bmatrix} K \\ -3K \end{bmatrix} \Rightarrow$ The simplest eigenvector is $X_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

For $\lambda = 5 \Rightarrow \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 5 \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$

$4X_1 + X_2 = 5X_1 \Rightarrow X_1 = X_2 \quad \therefore X_2 = \begin{bmatrix} K \\ K \end{bmatrix} \Rightarrow$ solution is $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

HW/ Determine The eigenvalues and eigenvectors if

$$A = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 4 & -1 \\ -1 & 2 & 0 \end{bmatrix}$$

Ans/ $\lambda = 1, 2, 3$

$X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

HW/ Find The eigenvectors of $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ -2 & 1 & -1 \end{bmatrix}$

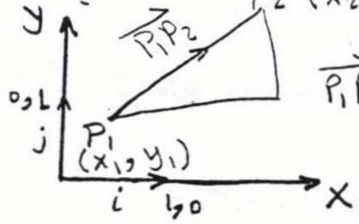
Ans/ $X_1 = \begin{bmatrix} 1 \\ 2 \\ -7 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

Vectors

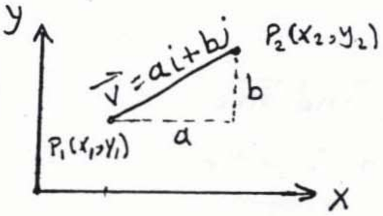
* Scalar & vector quantities:

- Scalar quantities have magnitude only (length, mass, time, area, volume, ... etc)
- vector quantities have magnitude & direction (force, velocity, acceleration).

So, a speed of 10km/h is a scalar quantity, but a velocity of 10km/h due North is a vector quantity.



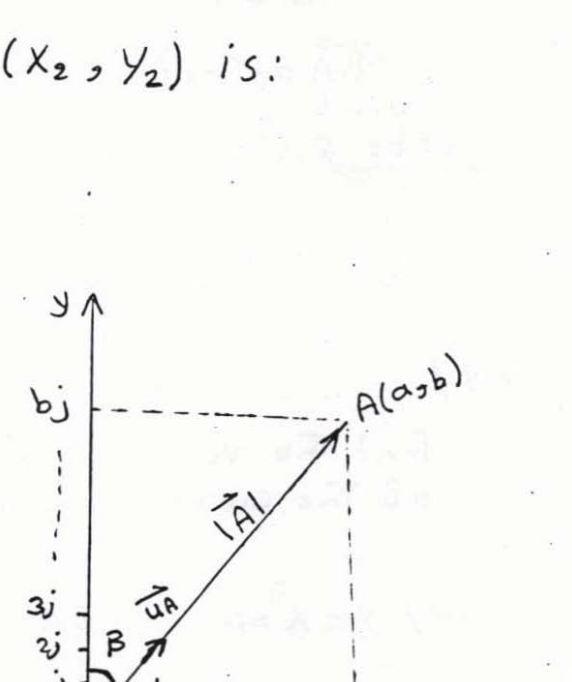
$\overrightarrow{P_1P_2} = (x_2 - x_1)i + (y_2 - y_1)j$



The vector between $P_1(x_1, y_1)$ & $P_2(x_2, y_2)$ is:

$\overrightarrow{P_1P_2} = (x_2 - x_1)i + (y_2 - y_1)j = \vec{v}$

$\overrightarrow{P_2P_1} = -\overrightarrow{P_1P_2} = (x_1 - x_2)i + (y_1 - y_2)j$



From Figure

$\alpha + \beta = \frac{\pi}{2}$

$\cos\alpha = \sin\beta$ & $\cos\beta = \sin\alpha$

$\vec{A} = \overrightarrow{OA} = ai + bj$

$|\vec{A}| = \sqrt{a^2 + b^2}$ = length

α, β : direction angles.
a, b : direction numbers.

$\cos\alpha = \frac{a}{|\vec{A}|}$, $\cos\beta = \frac{b}{|\vec{A}|}$ defined as direction cosine.

* unit vectors $= \overrightarrow{UA} = \dfrac{\overrightarrow{A}}{|\overrightarrow{A}|} = \dfrac{ai+bj}{\sqrt{a^2+b^2}} = \dfrac{a}{\sqrt{a^2+b^2}} i + \dfrac{b}{\sqrt{a^2+b^2}} j$

$= \cos\alpha\, i + \cos\beta\, j = \cos\alpha\, i + \sin\alpha\, j = \sin\beta\, i + \cos\beta\, j$

$|\overrightarrow{UA}| = 1$ --- prove That

EX/ Find unit vector which make $30°$ with The X-axis.

Sol/ $\overrightarrow{UA} = \cos\alpha\, i + \sin\alpha\, j \Rightarrow \alpha = 30°$

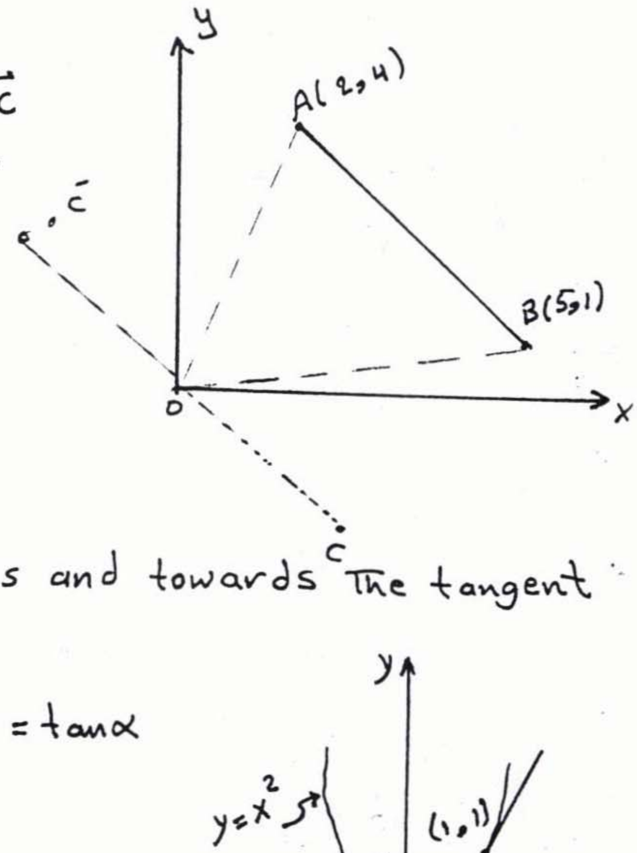
$= \frac{\sqrt{3}}{2} i + \frac{1}{2} j$

EX/ Find The vector towerds The line which pass Through $A(2,4)$, $B(5,1)$.

Sol/ $\overrightarrow{AB} = (5-2)i + (1-4)j = 3i - 3j = \overrightarrow{OC}$

$\overrightarrow{BA} = (2-5)i + (4-1)j = -3i + 3j = \overrightarrow{OC'}$

Note $\overrightarrow{AB} = -\overrightarrow{BA}$



Ex/

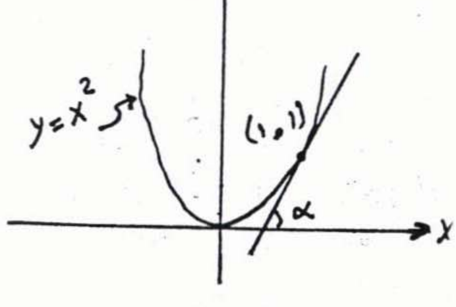
Find The vector with length 4 units and towards The tangent of The curve $y = x^2$ at $(1,1)$.

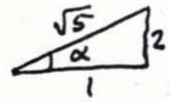
Sol/ $y = x^2 \Rightarrow \left.\dfrac{dy}{dx}\right|_{x=1} = 2x = 2 = m\,(\text{slope}) = \tan\alpha$

$\overrightarrow{u} = \cos\alpha\, i + \sin\alpha\, j = \frac{1}{\sqrt{5}} i + \frac{2}{\sqrt{5}} j$

$= \frac{1}{\sqrt{5}}(i + 2j)$

$\overrightarrow{A} = 4\overrightarrow{u} = \frac{4}{\sqrt{5}}(i + 2j)$.





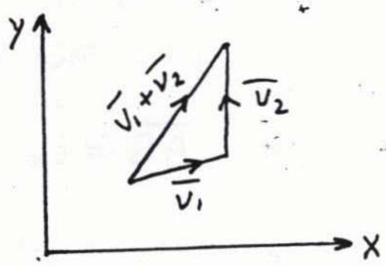
$\sin\alpha = 2/\sqrt{5}$

$\cos\alpha = 1/\sqrt{5}$

* vector operations

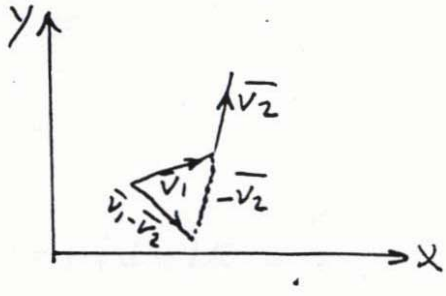
Ⓐ Addition of vectors

$$\left.\begin{array}{l}\vec{V_1} = a_1 i + a_2 j \\ \vec{V_2} = b_1 i + b_2 j\end{array}\right\} \vec{V_1} + \vec{V_2} = (a_1 + b_1) i + (a_2 + b_2) j$$

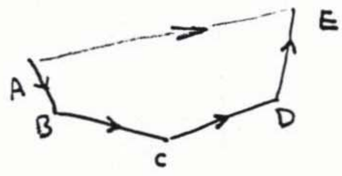


Ⓑ Subtraction of vectors

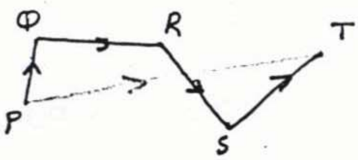
$$\vec{V_1} - \vec{V_2} = (a_1 - b_1) i + (a_2 - b_2) j$$



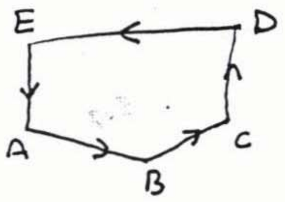
EX Find The sum of The vectors shown below:

① 

$$\overline{AB} + \overline{BC} + \overline{CD} + \overline{DE} = \overline{AE}$$

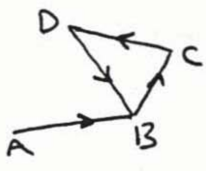
② 

$$\overline{PQ} + \overline{QR} + \overline{RS} + \overline{ST} = \overline{PT}$$

③ 

$$\overline{AB} + \overline{BC} + \overline{CD} + \overline{DE} + \overline{EA} = 0$$

④ $\overline{AB} + \overline{BC} - \overline{DC} - \overline{AD} = \overline{AB} + \overline{BC} + \overline{CD} + \overline{DA} = 0$

⑤ Find The resultant of 

$\equiv \overline{AB}$

Ⓒ multiplication by scalar : If c is scalar, & $\vec{V} = ai + bj$ The

$$c\vec{V} = aci + bcj$$

EX/ $c = 3$, $\overline{V} = 2i + 3j$

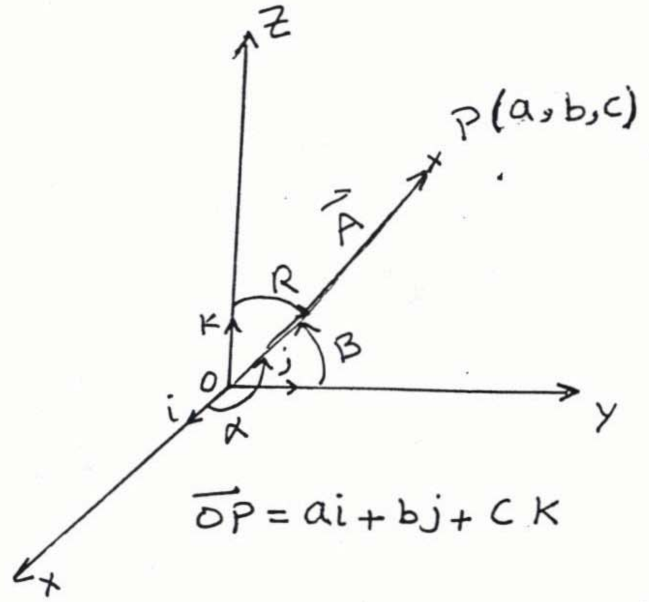
$$c\overline{V} = 6i + 9j$$

Ex/ Find The Sum of the two vectors $\overrightarrow{AB}$ & $\overrightarrow{CD}$ given The points $A(1,-1)$, $B(2,0)$, $C(-1,3)$, & $D(-2,2)$

Sol/ $\overrightarrow{AB} = i+j$, $\overrightarrow{CD} = -i-j \Rightarrow \overrightarrow{AB} + \overrightarrow{CD} = 0$

* Vector in Space

i = unit vector in x- direction.
j = " " " y- " .
k = " " " z- " .



$\overrightarrow{OP} = ai + bj + ck$

$$\vec{A} = ai+bj+ck$$

$$|\vec{A}| = \sqrt{a^2+b^2+c^2}$$

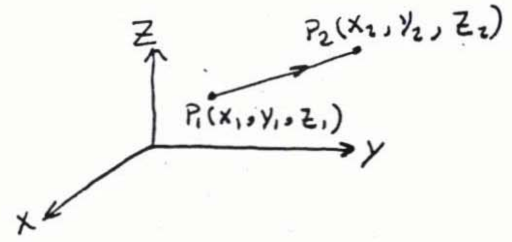
$$\overrightarrow{U_A} = \frac{\vec{A}}{|\vec{A}|} = \frac{ai+bj+ck}{\sqrt{a^2+b^2+c^2}}$$

$$= \frac{a}{|\vec{A}|} i + \frac{b}{|\vec{A}|} j + \frac{c}{|\vec{A}|} k$$

$$= \cos\alpha\, i + \cos B\, j + \cos R\, k$$

$$\cos^2\alpha + \cos^2 B + \cos^2 R = 1$$

* Distance between two points



$$\overline{P_1P_2} = (x_2-x_1)i + (y_2-y_1)j + (z_2-z_1)k$$

$$d = \text{distance} = |\overline{P_1P_2}|$$

$$= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

EX / ① Find a unit vector having The same direction of The vector $4i+3j+12K$.

② The distance from The point $P(x,y,z)$ to The origin is d_1 & The distance from P to $A(0,0,3)$ is d_2. Find The Locus of P if $d_1 = 2d_2$

sol/ ① $\bar{V} = 4i+3j+12K \Rightarrow |\bar{V}| = \sqrt{16+9+144} = 13$

$\therefore \vec{u} = \frac{4}{13}i + \frac{3}{13}j + \frac{12}{13}K$

② $d_1 = \sqrt{x^2+y^2+z^2}$, $d_2 = \sqrt{x^2+y^2+(z-3)^2}$

$\because d_1 = 2d_2$

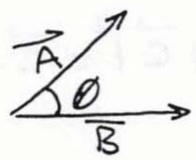
$x^2+y^2+z^2 = 4(x^2+y^2+z^2-6z+9)$

$x^2+y^2+(z-4)^2 = 4$

$\therefore$ sphere center at $(0,0,4)$ & radius $\sqrt{4} = 2$

* scalar product of two vectors $\vec{A}$ & $\bar{B}$;

(Dot product)



$\bar{A} = a_1 i + b_1 j + c_1 K$

$\bar{B} = a_2 i + b_2 j + c_2 K$

$i.i = j.j = K.K = 1$

$i.j = i.K = j.K = 0$

- $\vec{A}\cdot\vec{B} = |\vec{A}|\cdot|\vec{B}|\cdot \cos\phi$
- $\vec{A}\cdot\vec{B} = a_1a_2 + b_1b_2 + c_1c_2$

Note

① $\bar{A}.\bar{B} = \bar{B}.\bar{A}$

② $\cos\phi = \frac{\bar{A}.\bar{B}}{|\bar{A}||\bar{B}|}$

③ $\bar{A}\cdot\bar{B} = 0$ If $\vec{A} \perp \vec{B}$

④ $\vec{A}.\vec{A} = |\vec{A}|^2$

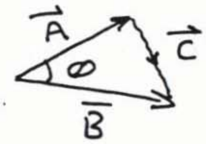
* Theorm

If $\vec{A} = a_1 i + a_2 j + a_3 k$, $\vec{B} = b_1 i + b_2 j + b_3 k$ are two vectors & ϕ is The angle between $\vec{A}$ & $\vec{B}$ Then

$$\vec{A} \cdot \vec{B} = a_1 b_1 + a_2 b_2 + a_3 b_3 = |\vec{A}||\vec{B}| \cos\phi$$

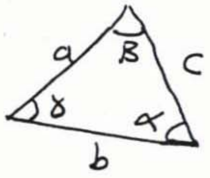
Prove

① $|\vec{C}|^2 = |\vec{A}|^2 + |\vec{B}|^2 - 2|\vec{A}||\vec{B}| \cos\phi$



note

- $c^2 = a^2 + b^2 - 2ab \cos\gamma$
- $a^2 = b^2 + c^2 - 2bc \cos\alpha$
- $b^2 = a^2 + c^2 - 2ac \cos\beta$
- $\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta} = \dfrac{c}{\sin\gamma}$



② $|\vec{C}|^2 = |\vec{B} - \vec{A}|^2$

③ $|\vec{C}|^2 = \vec{C} \cdot \vec{C}$

HW/ ① Find The angle between two vectors

$\vec{A} = 2i + 3j + k$ & $\vec{B} = i - 3j + 2k$ Ans/ $\phi = 110^\circ 55'$

② Find The vector normal to The vector $A = 2i - 3j$

Ans/ $\vec{N} = 3i + 2j$

Note The normal vector to The vector $\vec{A} = bi - aj$ is $\vec{N} = ai + bj$

③ Find The normal vector to The vector $2i - 3j + 4k$

Sol/ There is ∞ number of vectors (in space).