22.1 Simplex Method

Let us consider the 3 x 3 matrix

<table>
<thead>
<tr>
<th></th>
<th>B1</th>
<th>B2</th>
<th>B3</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1</td>
<td>a11</td>
<td>a12</td>
<td>a13</td>
</tr>
<tr>
<td>A2</td>
<td>a21</td>
<td>a22</td>
<td>a23</td>
</tr>
<tr>
<td>A3</td>
<td>a31</td>
<td>a32</td>
<td>a33</td>
</tr>
</tbody>
</table>

As per the assumptions, A always attempts to choose the set of strategies with the non-zero probabilities say \( p_1, p_2, p_3 \) where \( p_1 + p_2 + p_3 = 1 \) that maximizes his minimum expected gain.

Similarly B would choose the set of strategies with the non-zero probabilities say \( q_1, q_2, q_3 \) where \( q_1 + q_2 + q_3 = 1 \) that minimizes his maximum expected loss.

**Step 1**
Find the minimax and maximin value from the given matrix

**Step 2**
The objective of A is to maximize the value, which is equivalent to minimizing the value \( 1/V \). The LPP is written as

\[
\text{Min } 1/V = \frac{p_1}{V} + \frac{p_2}{V} + \frac{p_3}{V}
\]

and constraints \( \geq 1 \)

It is written as

\[
\text{Min } 1/V = x_1 + x_2 + x_3
\]

and constraints \( \geq 1 \)

Similarly for B, we get the LPP as the dual of the above LPP

\[
\text{Max } 1/V = Y_1 + Y_2 + Y_3
\]

and constraints \( \leq 1 \)

Where \( Y_1 = q_1/V, Y_2 = q_2/V, Y_3 = q_3/V \)

**Step 3**
Solve the LPP by using simplex table and obtain the best strategy for the players
Example 1
Solve by Simplex method

\[
\begin{bmatrix}
3 & -2 & 4 \\
-1 & 4 & 2 \\
2 & 2 & 6
\end{bmatrix}
\]

Solution

\[
\begin{bmatrix}
3 & -2 & 4 \\
-1 & 4 & 2 \\
2 & 2 & 6
\end{bmatrix}
\]

We can infer that \( 2 \leq V \leq 3 \). Hence it can be concluded that the value of the game lies between 2 and 3 and the \( V > 0 \).

LPP
Max \( \frac{1}{V} = Y_1 + Y_2 + Y_3 \)
Subject to
\[
\begin{align*}
3Y_1 - 2Y_2 + 4Y_3 & \leq 1 \\
-1Y_1 + 4Y_2 + 2Y_3 & \leq 1 \\
2Y_1 + 2Y_2 + 6Y_3 & \leq 1 \\
Y_1, Y_2, Y_3 & \geq 0
\end{align*}
\]

SLPP
Max \( \frac{1}{V} = Y_1 + Y_2 + Y_3 + 0s_1 + 0s_2 + 0s_3 \)
Subject to
\[
\begin{align*}
3Y_1 - 2Y_2 + 4Y_3 + s_1 & = 1 \\
-1Y_1 + 4Y_2 + 2Y_3 + s_2 & = 1 \\
2Y_1 + 2Y_2 + 6Y_3 + s_3 & = 1 \\
Y_1, Y_2, Y_3, s_1, s_2, s_3 & \geq 0
\end{align*}
\]
$$\begin{array}{c|cccccccc}
C_j & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline
\text{Basic Variables} & C_B & Y_B & Y_1 & Y_2 & Y_3 & S_1 & S_2 & S_3 & \text{Min Ratio} \\
S_1 & 0 & 1 & 3 & -2 & 4 & 1 & 0 & 0 & 1/3 \rightarrow \\
S_2 & 0 & 1 & -1 & 4 & 2 & 0 & 1 & 0 & - \\
S_3 & 0 & 1 & 2 & 2 & 6 & 0 & 0 & 1 & 1/2 \\
\hline
1/V = 0 & \uparrow & \uparrow & -1 & -1 & -1 & 0 & 0 & 0 \text{ } 1/3 \\
Y_1 & 1 & 1/3 & 1 & -2/3 & 4/3 & 1/3 & 0 & 0 & - \\
S_2 & 0 & 4/3 & 0 & 10/3 & 10/3 & 1/3 & 1 & 0 & 2/5 \\
S_3 & 0 & 1/3 & 0 & \text{10/3} & 10/3 & -2/3 & 0 & 1 & 1/10 \rightarrow \\
\hline
1/V = 1/3 & \uparrow & \uparrow & \uparrow & 0 & 0 & 0 & 0 & 0 & - \\
Y_1 & 1 & 2/5 & 1 & 0 & 2 & 1/5 & 0 & 1/5 \\
S_2 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & -1 \\
Y_2 & 1 & 1/10 & 0 & 1 & 1 & -1/5 & 0 & 3/10 \\
\hline
1/V = 1/2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1/2 \\
\end{array}$$

1/V = 1/2

V = 2

\(y_1 = \frac{2}{5} \times 2 = \frac{4}{5}\)

\(y_2 = \frac{1}{10} \times 2 = \frac{1}{5}\)

\(y_3 = 0 \times 2 = 0\)

\(x_1 = 0 \times 2 = 0\)

\(x_2 = 0 \times 2 = 0\)

\(x_3 = \frac{1}{2} \times 2 = 1\)

\(S_A = (0, 0, 1)\)

\(S_B = (4/5, 1/5, 0)\)

Value = 2

**Example 2**

$$A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & -1 & 3 \\ -1 & 2 & -1 \end{bmatrix}$$
Solution

\[
\begin{bmatrix}
1 & -1 & -1 \\
-1 & -1 & 3 \\
-1 & 2 & -1 \\
1 & 2 & 3
\end{bmatrix}
\]

Maximin = -1
Minimax = 1

We can infer that \(-1 \leq V \leq 1\)

Since maximin value is -1, it is possible that value of the game may be negative or zero, thus the constant ‘C’ is added to all the elements of matrix which is at least equal to the negative of maximin.

Let C = 1, add this value to all the elements of the matrix. The resultant matrix is

\[
\begin{bmatrix}
2 & 0 & 0 \\
0 & 0 & 4 \\
0 & 3 & 0
\end{bmatrix}
\]

LPP
Max \(\frac{1}{V} = Y_1 + Y_2 + Y_3\)
Subject to
\[
\begin{align*}
2Y_1 + 0Y_2 + 0Y_3 & \leq 1 \\
0Y_1 + 0Y_2 + 4Y_3 & \leq 1 \\
0Y_1 + 3Y_2 + 0Y_3 & \leq 1 \\
Y_1, Y_2, Y_3 & \geq 0
\end{align*}
\]

SLPP
Max \(\frac{1}{V} = Y_1 + Y_2 + Y_3 + 0s_1 + 0s_2 + 0s_3\)
Subject to
\[
\begin{align*}
2Y_1 + 0Y_2 + 0Y_3 + s_1 & = 1 \\
0Y_1 + 0Y_2 + 4Y_3 + s_2 & = 1 \\
0Y_1 + 3Y_2 + 0Y_3 + s_3 & = 1 \\
Y_1, Y_2, Y_3, s_1, s_2, s_3 & \geq 0
\end{align*}
\]
<table>
<thead>
<tr>
<th>Basic Variables</th>
<th>$C_B$</th>
<th>$Y_B$</th>
<th>$Y_1$</th>
<th>$Y_2$</th>
<th>$Y_3$</th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$S_3$</th>
<th>Min Ratio $Y_B / Y_K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1/2</td>
</tr>
<tr>
<td>$S_2$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>$S_3$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

$1/V = 0$

| $Y_1$          | 1    | 1/2  | 1    | 0    | 0    | 1/2  | 0    | 0    | -                |
| $S_2$          | 0    | 1    | 0    | 0    | 4    | 0    | 1    | 0    | 1/4              |
| $S_3$          | 0    | 1    | 0    | 3    | 0    | 0    | 0    | 1    | 1/3              |

$1/V = 1/2$

| $Y_1$          | 1    | 1/2  | 1    | 0    | 0    | 1/2  | 0    | 0    | -                |
| $S_2$          | 0    | 1    | 0    | 0    | 4    | 0    | 1    | 0    | 1/4              |
| $S_3$          | 0    | 1    | 0    | 3    | 0    | 0    | 0    | 1    | 1/3              |

$1/V = 5/6$

| $Y_1$          | 1    | 1/2  | 1    | 0    | 0    | 1/2  | 0    | 0    | -                |
| $Y_3$          | 1    | 1/4  | 0    | 0    | 1    | 0    | 1/4  | 0    |                  |
| $Y_2$          | 1    | 1/3  | 0    | 1    | 0    | 0    | 0    | 1/3             |

$1/V = 13/12$

$1/V = 13/12$

$V = 12/13$

$y_1 = 1/2 \times 12/13 = 6/13$

$y_2 = 1/3 \times 12/13 = 4/13$

$y_3 = 1/4 \times 12/13 = 3/13$

$x_1 = 1/2 \times 12/13 = 6/13$

$x_2 = 1/4 \times 12/13 = 3/13$

$x_3 = 1/3 \times 12/13 = 4/13$

$S_A = (6/13, 3/13, 4/13)$

$S_B = (6/13, 4/13, 3/13)$

Value $= 12/13 - C = 12/13 - 1 = -1/13$