# Chapter Two

Yield line

Yield line analysis for slabs

Most concrete slabs are designed for moments found by the methods based essentially upon elastic theory. On the other hand, reinforcement for slabs is calculated by strength methods. That account for the actual inelastic behavior of member at the factored load stage. A corresponding contradiction exists in the process by which beams and frames are analyzed and designed, and the concept of limit, or plastic analysis of reinforced concrete was introduced.

For slabs there is a good reason for interest in limit analysis. The elasticity-based methods are restricted in important ways. But in the practice, many slabs do not meet these restrictions, for example for round or triangular slabs, slabs with large openings, slabs supported on two or three edges only (as shown in fig. below), and slabs carrying concentrated loads. Limit analysis provides a powerful tool for treating such problems.
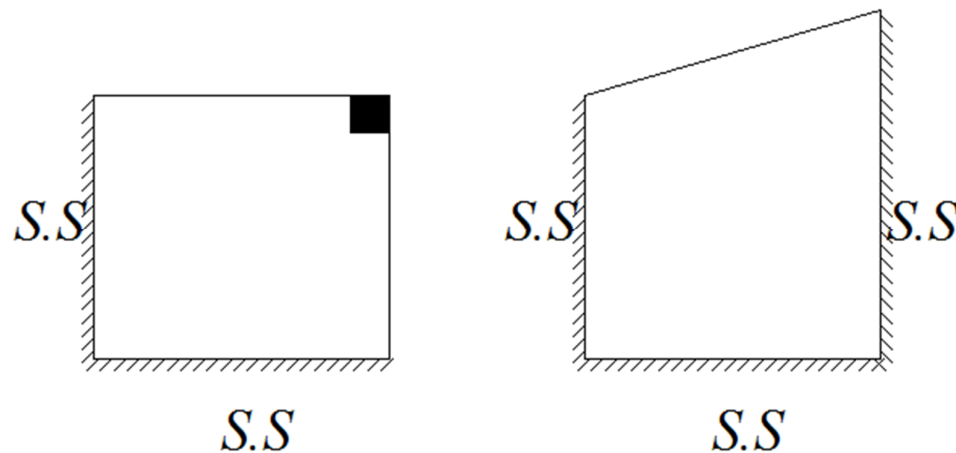
Fig(2-): Slabs supported at two edges or three.

For slabs which typically have tensile reinforcement ratio much below the balanced value and consequently have large rotation capacity it can be safely assumed that the necessary ductility is present. Practical methods for the plastic analysis are thus possible and Yield line theory is one of these.

The plastic hinge was defined as a location along a member in a continuance beam or frame at which upon over loading, there would be large inelastic rotation at essentially a constant resisting moment. For slabs

the corresponding mechanism is the yield line. The yield line serves as an axis of rotation for the slab segment.

A simply supported uniformly loaded reinforced concrete slab shown in Fig(2-2) with $\rho < \rho_b$. The elastic moment is shown, as the load is increased, when the applied moment becomes equal to the flexural capacity of the slab cross section. The tensile steel starts to yield along the transvers line of max moment.

Upon yielding, the curvature of the slab at the yielding section increases sharply. And deflection increases disproportionately.

The hinge that forms at the yield line rotates with essentially constant resistance.

The resistance (m_p) per unit width of the slab is the nominal flexural strength of the slab

$$m_p = m_n$$

For design purposes, m_p will be taken equal to $\emptyset m_n$

$$M_{el} = \frac{wL^2}{8}$$

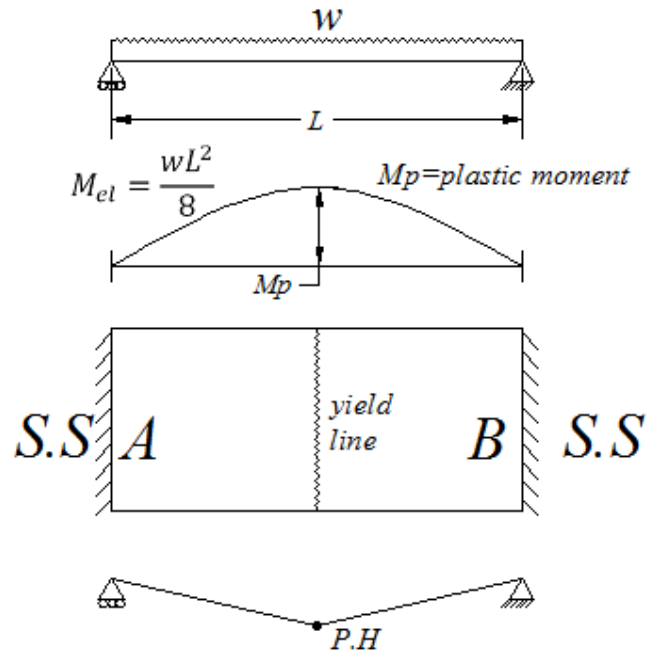Mp=plastic moment

Mp

S.S A   yield line   B   S.S

P.H

Fig.(2-2) Slab simply supported uniformly loaded one way slab
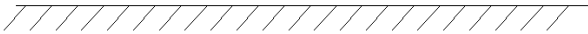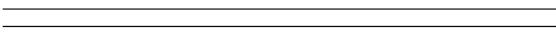
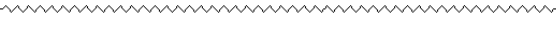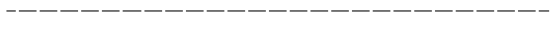The fixed-fixed slab shown in Fig(2-3) is loaded uniformly and assumed to be equally reinforced for (+m) and (−m). As the load in increased the more hingly stressed section at support start yielding. Rotations occur at the support line hinges. The load can be increased further, until the moment at mid span becomes equal to the moment capacity there, and a third yield forms. The slab is now a mechanism. Large deflection occurs and collapse takes place. Elastic distribution of moments shown in Fig (2-2 b) the ratio of these moments just before collapse (Fig 2-2 c)

$M_{el} = \dfrac{wL^2}{24}$

$-M = \dfrac{wL^2}{12}$

$-M = \dfrac{wL^2}{12}$

$+M = -M = \dfrac{wl^2}{16}$
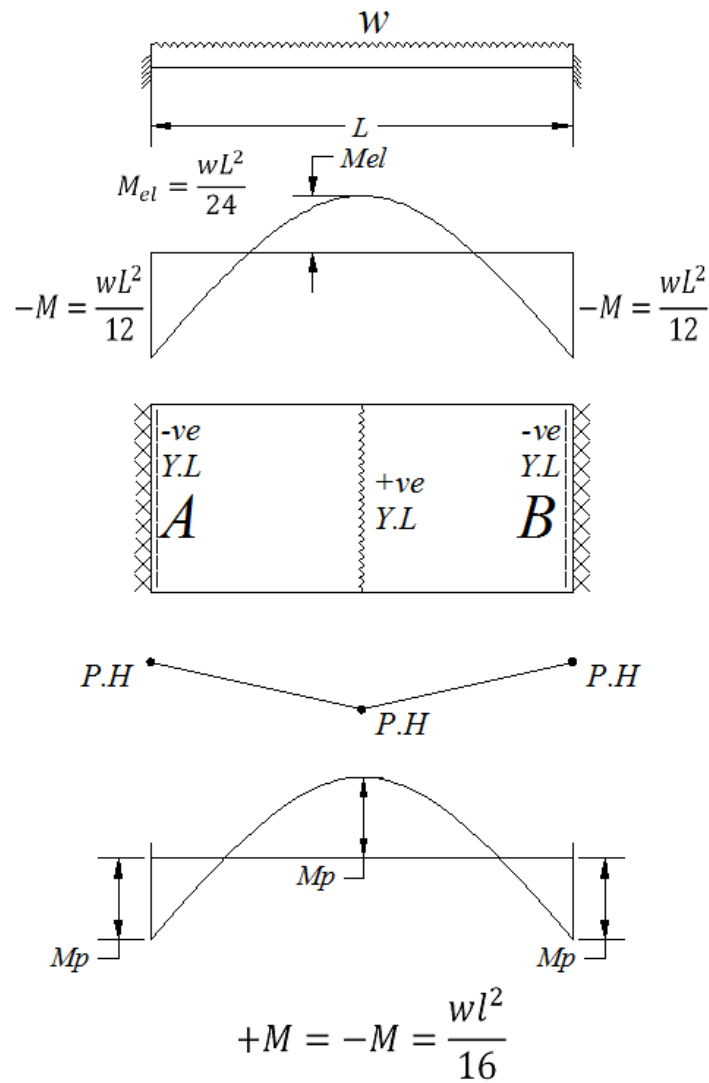
Fig(2-2) Fixed-end uniformly loaded one-way slab

The terms positive yield line and negative yield line are used to distinguish between those associated with tension at the bottom and tension at the top of the slab respectively

Notation

■	Column
/////////////////////////	Simply supported
XXXXXXXXXXXXXXXXXXXXXX	Either cautious of fixed end
═══════════════════	Beam
~~	+ve Y.L [Tension at bottom face]
– – – – – – – – – – – – – – – –	-ve Y.L [Tension at Top face]
○	Point
— · — · — · — · — · — · —	Axes of rotation

Rules of yield lines

1- Yield lines are straight lines because they represent the intersection of two planes.
2- Yield lines represent axes of rotation.
3- The support edges of the slab will also establish axes of rotation.
4- An axis of rotation will pass over any column.
5- Yield lines from under concentrated load radiating out word from the point of application.
6- A yield line between two slab segments must pass through the point of intersection of the axes of rotation of the adjacent slab system.

In Fig 2-3 which shows a slab supported along its four sides, rotation of slab segments A and B is about ab and cd respectively. The yield line ef between these two segments is a straight line passing through f the point of intersection of the axes of rotation.

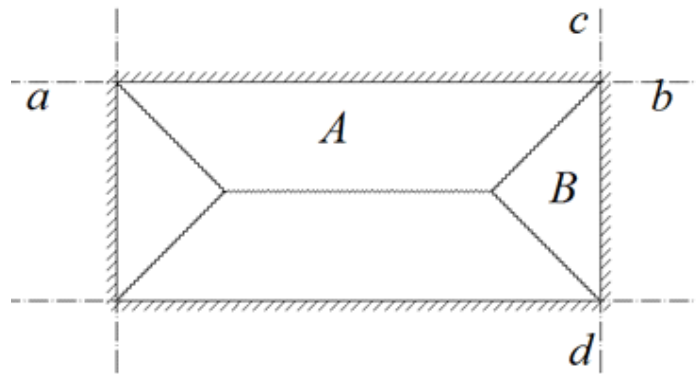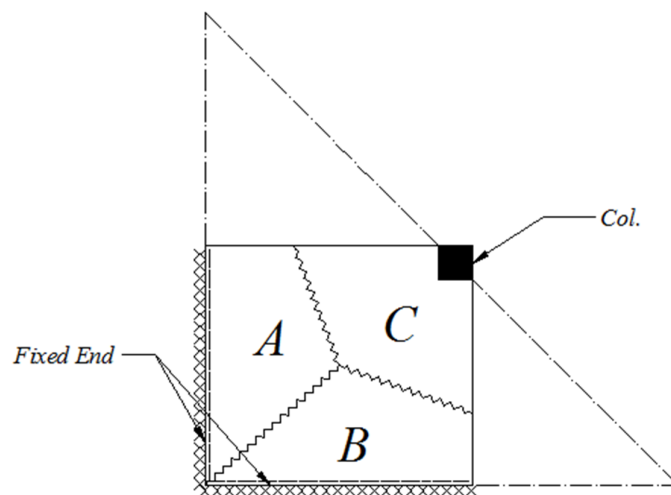Fig.(2-3) Two-way slab with simply supported edges

Simply supported all sides

(a)

(c)

S.S

S.S

(b)

Fixed End

(d)

Col.

Fixed End

A   C

B

(f)

Free edge

four Column
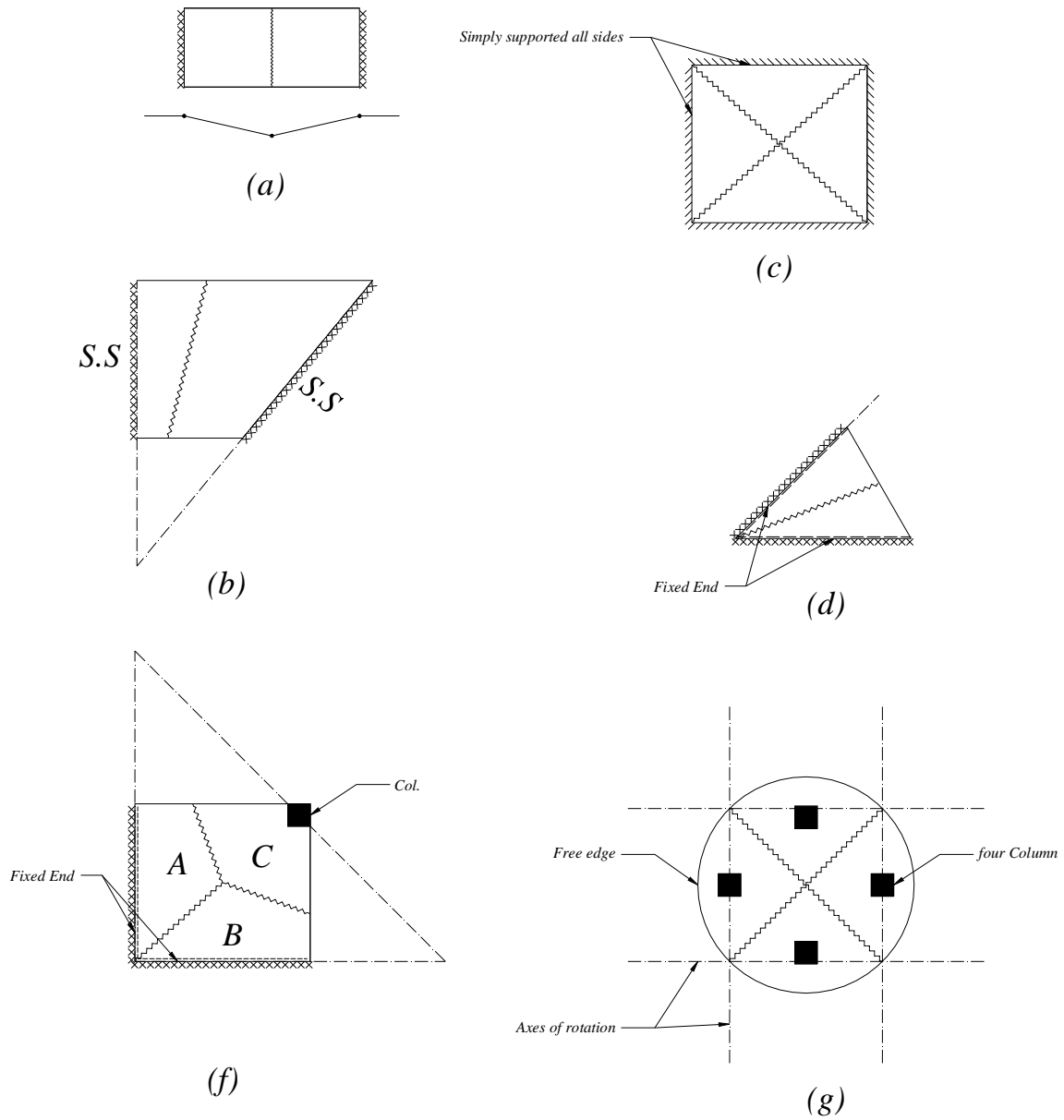
Axes of rotation

(g)

Fig.(2-4) Typical yield line patterns

Method of analysis for yield line

   1- Method of segment equilibrium

      It requires consideration of the equilibrium of the individual slab segments forming the collapse mechanism and leads to a set of simultaneous equations permitting solution for the unknown geometric parameters and for the relation between load capacity and resisting moments.

   2- Method of virtual work

      This method is based on equating the internal work done at the plastic hinges with the external work done by the loads as the predefined failure mechanism is given a small virtual displacement.

      The yield line method of analysis for slabs is an upper bound approach in the sense that the true collapse load will never be higher, but may be lower, and then the load predicted. The solution has two essential parts:

      a- Establishing the correct failure pattern

      b- Finding the geometric parameters that define the exact location and orientation of the yield lines and solving for the relation between applied load and resisting moments.

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Eternal work done by load

An external load acting on a slab segment, as a small virtual displacement is imposed does work equal to the product of its constant magnitude and the distance through which the point of application of the load moves. If the load is distributed over a length or an area, rather than concentrated, the work can be the work can be calculated as the product of the total load and the displacement of the point of application of its resultant.

- If a segments slab carrying a single concentrated load at its center (Fig(2-5 a)). Is given a virtual displacement defined by a unit value under the load, the external work is

$$W_e = P * 1.0$$

- When a distributed load (w) per unit area acts on a triangular segment defined by a hinge and yield lines, such as Fig.(2-5b)

$$W_e = \frac{w\,a\,b}{2} * \frac{1}{3} = \frac{w\,a\,b}{6}$$

While for the rectangular slab segment shown in Fig(2-5 c). Carrying a distributed load w per unit area, the external work is

$$W_e = \frac{w\,a\,b}{2}$$

- More complicated trapezoidal shapes may always be subdivided into component triangles and rectangles
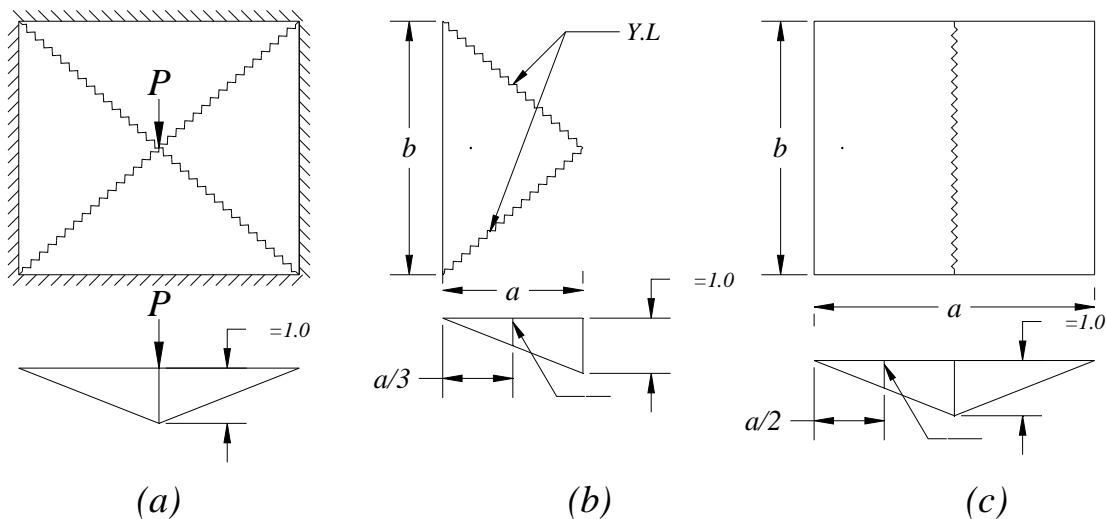
Fig(2-5) External work basis for various types of loads.

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Internal work done by resisting moments:

The internal work done during the assigned virtual displacement is found by summing the production of yield moment (m) per unit length of hinge times the plastic rotation (θ) at the respective yield lines. If the resisting moment (m) is constant along a yield line of length (L) and if a rotation (θ) is experienced, the internal work is :

$$w_i = mL\emptyset \qquad\qquad L = Length\ of\ yield\ line$$

For the entire system, the total internal work done is the sum of the contributions from all yield lines.

Ex1: Determine the load capacity of the one-way uniformly loaded (w) simply supported slab shown in Fig. using the method of virtual work. The resisting moment of slab in (m).

Solution

$$W_e = w \propto L^2 * \frac{\delta}{2}$$

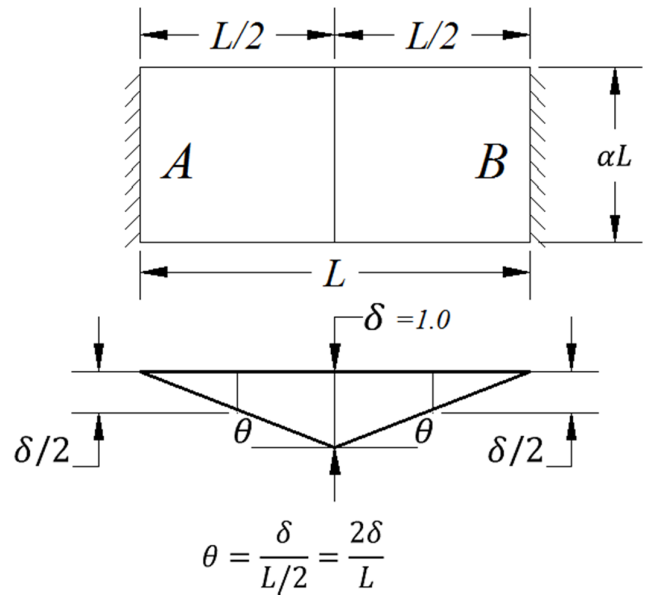$$W_e = w \propto L * \frac{L}{2} * \frac{\delta}{2} + w \propto L * \frac{L}{2} * \frac{\delta}{2}$$

$$W_i = 2m \propto L\emptyset = m \propto L\emptyset + m \propto L\emptyset$$

$$= 4m \propto \delta$$

$$W_e = W_i$$

$$w \propto L^2 * \frac{\delta}{2} = 4m \propto \delta$$

$$m = \frac{wL^2}{8}$$

$$\theta = \frac{\delta}{L/2} = \frac{2\delta}{L}$$

Ex

$$w_e = w * (\propto L * \lambda L) * \frac{\delta}{2} + w *\propto L * (L - \lambda L) * \frac{\delta}{2} = w \propto L^2 \frac{\delta}{2}$$

$$\theta_1 = \frac{\delta}{\lambda L} \quad , \quad \theta_2 = \frac{\delta}{L - \lambda L}$$

$$w_i = m \propto L * \theta_1 + m \propto L\theta_2 = m \propto L(\theta_1 + \theta_2) = m$$

$$\propto L\left(\frac{\delta}{\lambda L} + \frac{\delta}{L - \lambda L}\right)$$

$$w_e = w_i$$
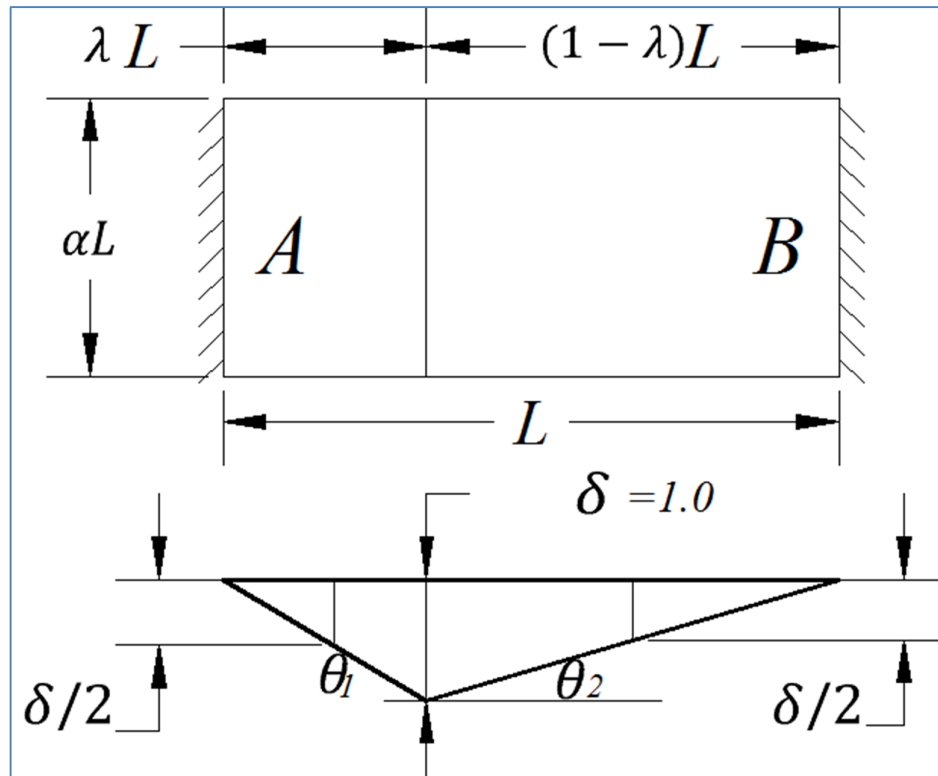
$$w \propto L^2 \frac{\delta}{2} = m \propto L\left(\frac{\delta}{\lambda L} + \frac{\delta}{L(1 - \lambda)}\right)$$

$$m = \frac{wL^2}{2}(1 - \lambda)\lambda$$

$$\frac{\partial m}{\partial \lambda} = 0 = \frac{wL^2}{4}(1 - 2\lambda)$$
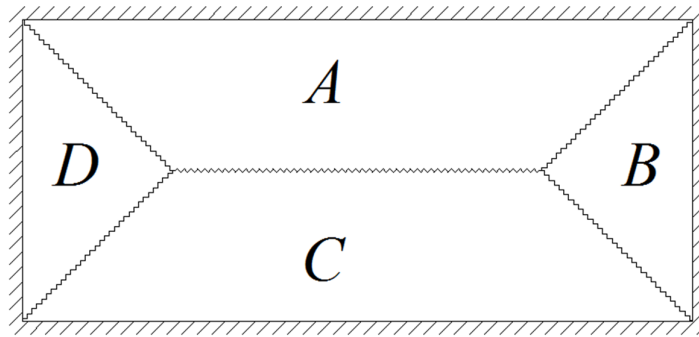
$$\lambda = \frac{1}{2}$$

$$m = \frac{wL^2}{8}$$

For two-way slab for

Orthotropic moment capacities (m_x, m_y)
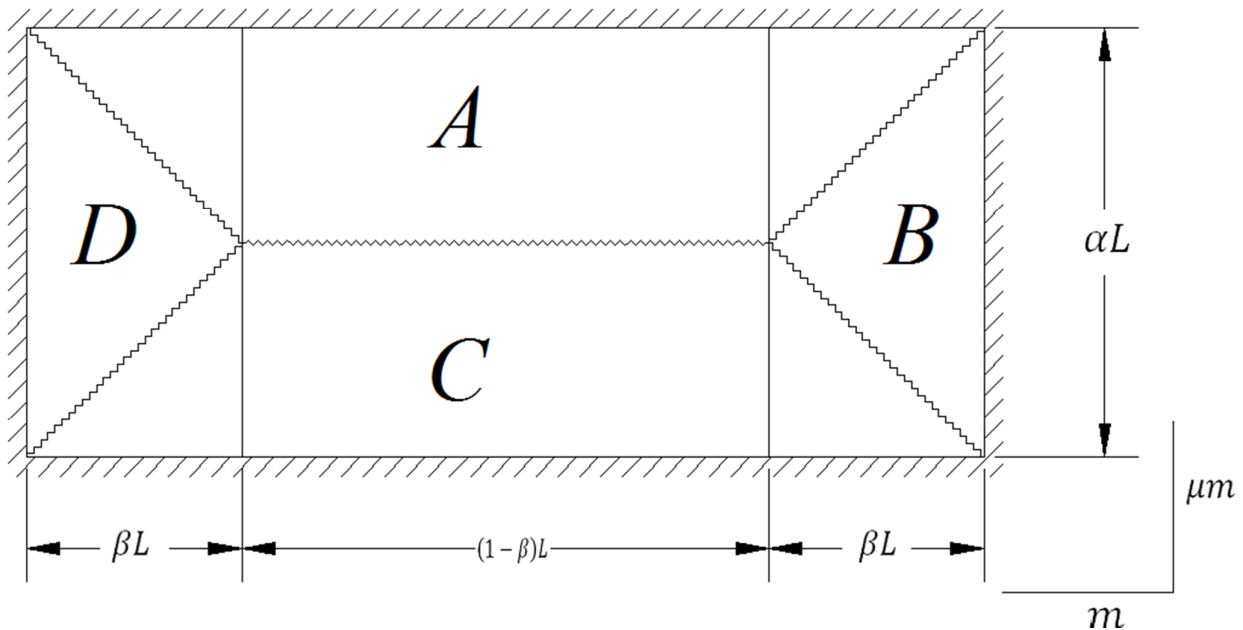
- Orthotropic
- Isotropic

For two-way slab orthotropic moment capacities (m_x, m_y) the internal work is

$$w_i = \Sigma\left(m_x * x * \theta_x + m_y * y * \theta_y\right)$$

Where m_x =ultimate moment per unit length in the direction of x-axis

Ex:-The slab shown is simply supported and it is orthotropic reinforced and loaded by uniformly distributed load find the moment capacity of the slab.

Solution

$$\sum M\theta_B = \sum(M_x * \theta_x + M_y * \theta_y) = \mu m * L * \frac{1}{\beta L}$$

$$\sum M\theta_A = \sum(M_x \theta_x + M_y \theta_y) = m * L * \frac{2}{\alpha L}$$

$$\sum M\theta_{A,B,C,D} =$$

$$\sum w\delta = \frac{w\alpha L}{2} * (1 - 2\beta) * L * \frac{1}{2} + \frac{w\alpha L}{2} * (1 - 2\beta) * L * \frac{1}{2}$$

$$+ \left(w\beta L * \frac{\alpha L}{2} * \frac{1}{3}\right) * 4 + \left(w * \frac{\alpha L\beta L}{2} * \frac{1}{3}\right) * 2$$

or $\sum w\delta = w\alpha L(1 - 2\beta)L * \dfrac{1}{2} + \left(w\alpha L * \beta L * \dfrac{1}{3}\right) * L$

$$= w\alpha L^2 \left(\frac{2}{3}\beta L + \frac{1 - 2\beta}{2}\right) \dots\dots\dots\dots\dots (II)$$

$$\sum M\theta = \sum w\delta$$

$$m = \frac{1}{12} * w * \alpha^2 L^2 * \left(\frac{3\beta - 2\beta^2}{2\beta + \mu\alpha^2}\right) \rightarrow \frac{\partial m}{\partial \beta} = 0$$

$$\frac{3\beta - 2\beta^2}{2\beta + \mu\alpha^2} = \frac{3 - 4\beta}{2}$$

$$4\beta^2 + 4\mu\alpha^2\beta - 3\mu\alpha^2 = 0$$

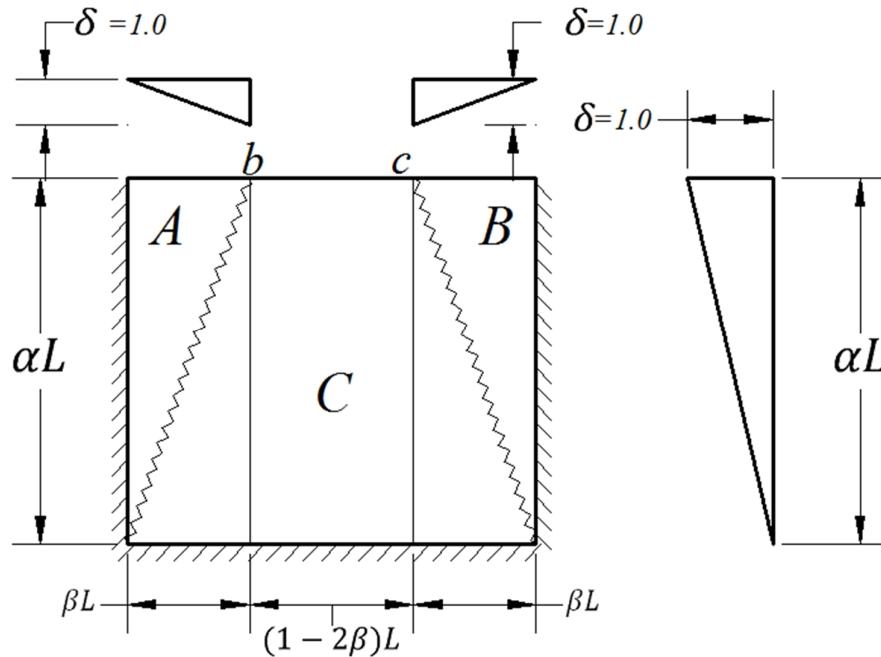$$\beta = \frac{1}{2}\left[\sqrt{3\mu\alpha^2 + \mu^2\alpha^4} - \mu\alpha^2\right]$$

$$\therefore m = \frac{1}{24} w\alpha^2 L^2 \left[\sqrt{3 + \mu\alpha^2} - \alpha\sqrt{\mu}\right]^2$$

Ex11:-

Using yield line theory of analysis fined by the method of virtual work the relationship between (m) and (w) for simply supported rectangular slab with orthotropic moment capacity ($\mu m, m$)

Sol

Case 1 (δ at b & c = 1)

$$\sum M\theta_{AB} = \sum(M_x * \theta_x + M_y\theta_y)\alpha L = \left(m\alpha L * \frac{1}{\beta L}\right) * 2$$

$$\sum M\theta_C = \sum(M_x\theta_x + M_y\theta_y) = \mu m * 2\beta L * \frac{1}{\alpha L}$$

$$\sum M\theta_{A,B,C} = 2m\left(\frac{\alpha}{\beta} + \frac{\mu\beta}{2}\right) \dots\dots (I)$$

$$\sum W\delta = w(1-2\beta)L * \alpha L * \frac{1}{2} + w * \frac{\beta L\alpha L}{2} * \frac{1}{3} * 2 + \frac{\beta L\alpha L}{2} * \frac{1}{3} * 2$$

$$\sum W\delta = \alpha wL^2\left(\frac{2\beta}{3} + \frac{1-2\beta}{2}\right) \dots\dots (II)$$
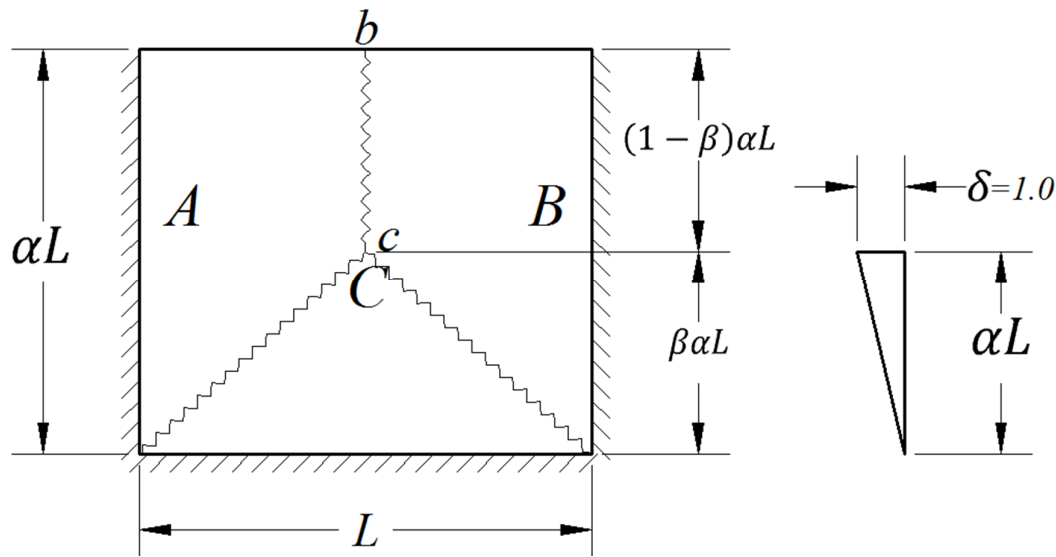
$$\sum M\theta = \sum w\delta$$

$$2m\left(\frac{\alpha}{\beta} + \frac{\mu\beta}{2}\right) = \alpha wL^2\left(\frac{2\beta}{3} + \frac{1-2\beta}{2}\right)$$

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$$m = \frac{\alpha w L^2}{12} * \left(\frac{3\beta\alpha * 2\beta^2\alpha}{\alpha^2 + \mu\beta^2}\right) \qquad \rightarrow \frac{\partial m}{\partial \beta} = 0$$

$$\beta = \frac{1}{6}\sqrt{\frac{4}{9} + \gamma} \qquad \gamma = \frac{\mu}{\alpha^2}$$

$$m = \frac{w\alpha^2 L^2}{24\mu}\left[\sqrt{4 + \frac{9\mu}{\alpha^2}} - 2\right]$$

Case 2 (δ at b & c = 1)

$$\sum M\theta_{A,B} = \sum\left(M_x * \theta_x + M_y\theta_y\right) = \left(m\alpha L * \frac{2}{L}\right) * 2$$

$$\sum M\theta_C = \sum\left(M_x * \theta_x + M_y\theta_y\right) = \mu m * L * \frac{1}{\beta\alpha L}$$

$$\sum M\theta_{A,B,C} = m\left(4\alpha + \frac{\mu}{\alpha\beta}\right) \ldots\ldots (I)$$

$$\sum w\delta = w(1-\beta)\alpha L^2 * \frac{1}{2} + w\beta\alpha L^2 * \frac{1}{3} = w\alpha L^2\left(\frac{1}{2} - \frac{\beta}{8}\right)$$

$$\sum M\theta = \sum w\delta$$

$$m = \frac{w\alpha L^2}{12}\left(\frac{6\alpha\beta - 2\alpha\beta^2}{4\alpha^2\beta + \mu}\right) \rightarrow \frac{\partial m}{\partial \beta} = 0$$

$$\beta = \frac{\gamma}{2}\left[\sqrt{\frac{1}{4} + \frac{3}{\gamma}} - \frac{1}{2}\right]$$

Ex:- for the reinforced concrete isotropic slab shown in Fig. find the relationship between the resistance moment (m) and uniform load (w) using yield line theory.
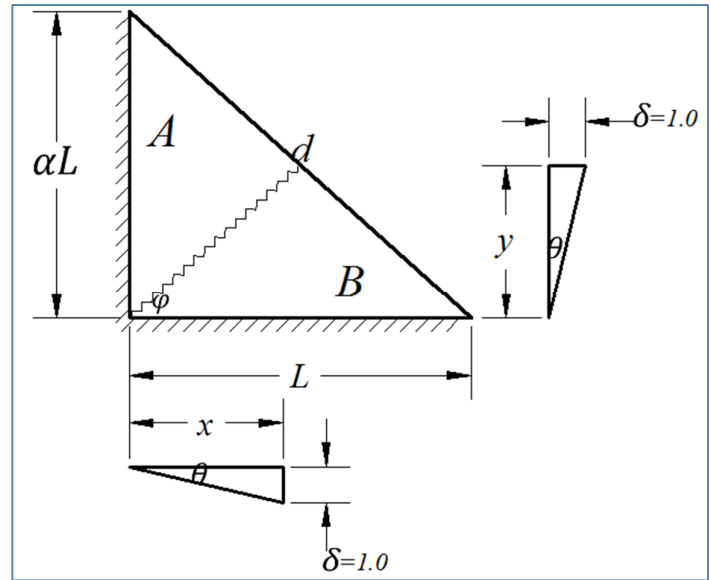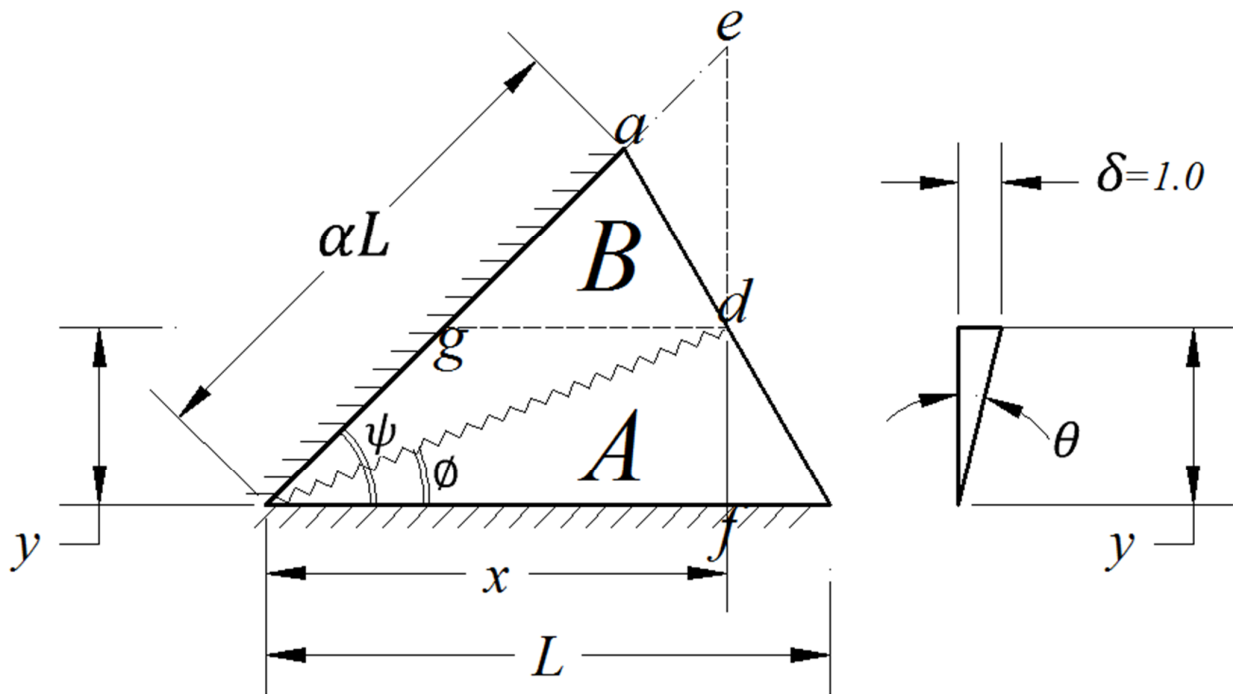
Sol:

(δ at $d = 1$)

$$\sum w\delta = \frac{L\alpha L}{2} * w * \frac{1}{3} = \frac{w\alpha L^2}{6}$$

$$\sum M\theta = m * x * \frac{1}{y} + m * y * \frac{1}{x}$$

$$= m\left(\frac{x}{y} + \frac{y}{x}\right)$$

$$\frac{x}{y} = \beta \rightarrow \beta = \pm 1 \therefore \varphi = 45^o$$

Ex:- using Y.L theory find the relationship between the moment of resistance of the isotropic slab (m) and uniform load (w) for the slab shown in Fig.

δ at $d \rightarrow 1$

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Solution

$$\sum M\theta_A = \sum(M_x * \theta_x + M_y\theta_y) = m * x * \frac{1}{y} = m * \frac{x}{y} = m\cot(\theta)$$

$$\sum M\theta_B = \sum(M_x * \theta_x + M_y\theta_y) = m * x * \frac{1}{de} + m * y * \frac{1}{dg}$$

$$de = ef - y = x\tan(\psi) - y = x\tan(\psi) - x\tan(\theta)$$
$$dg = x - y\cot(\psi) = y\cot(\theta) - y\cot(\psi) = y(\cot(\theta) - \cot(\psi))$$

$$\sum M\theta_B = m\left[\frac{1}{\tan(\psi) - \tan(\theta)} + \frac{1}{\cot(\theta) - \cot(\psi)}\right]$$

$$= m\left[\frac{1 + \tan(\psi)\cot(\theta)}{\tan(\psi) - \tan(\theta)}\right] = m[\cot(\psi - \theta)]$$

$$\sum M\theta_{A,B} = m(\cot(\theta) + \cot(\psi - \theta))$$

$$\sum w\delta = \frac{1}{3} * w\left[\frac{1}{2} * \alpha L * L * \sin(\psi)\right] = \frac{1}{6} * w * \alpha * L^2\sin(\psi)$$

$$\sum M\theta = \sum w\delta$$

$$m = \frac{w\alpha * L^2 * \sin(\psi)}{6[\cot(\psi - \theta) + \cot(\theta)]} = \frac{1}{6}w * \alpha * L^2 * \sin(\theta)\sin(\psi - \theta)$$

$$\frac{\partial m}{\partial \theta} = 0$$

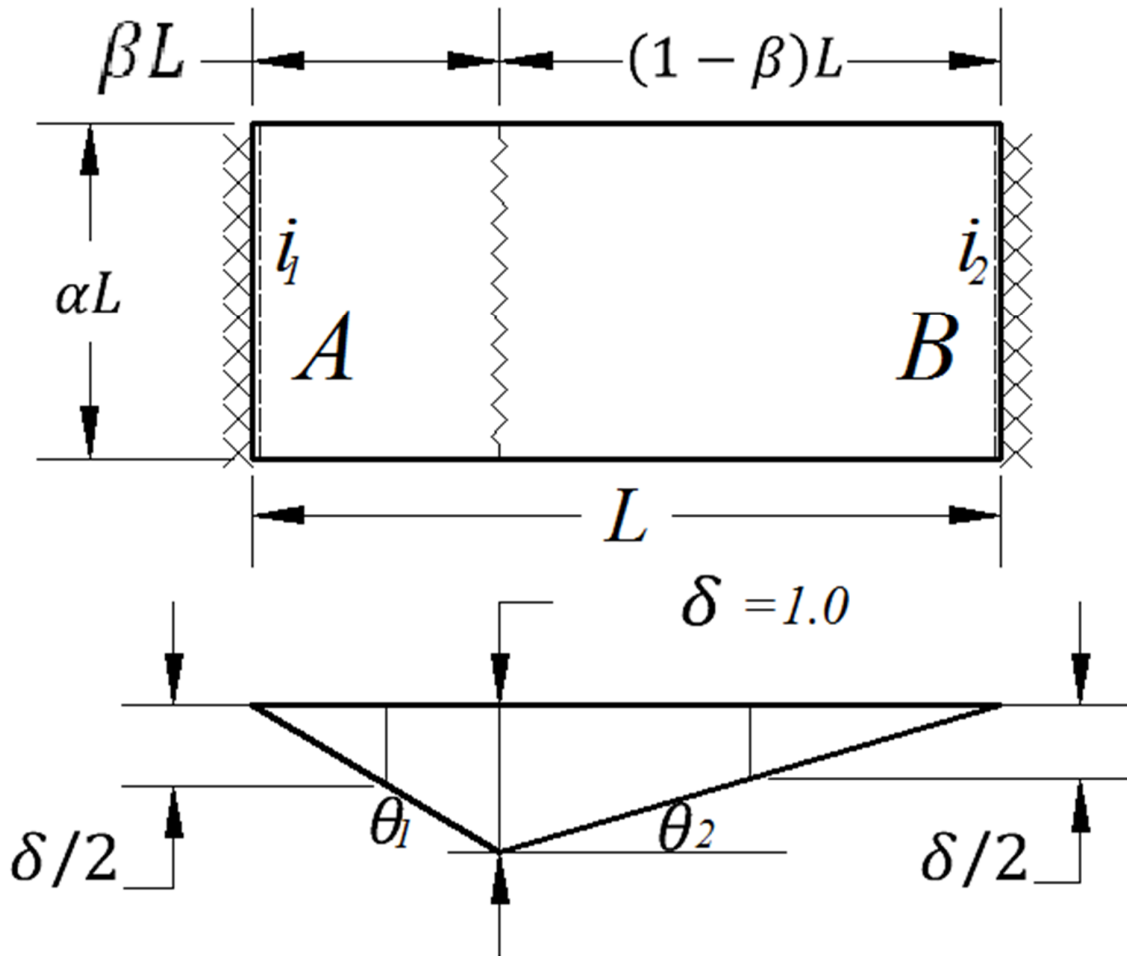$$\cos(\theta) * \sin(\psi - \theta) = \sin(\theta) * \cos(\psi - \theta)$$

$$\tan(\theta) = \tan(\psi - \theta)$$

$$\therefore \theta = \frac{1}{2}\psi$$

$$m = \frac{1}{6} * w * \alpha * L^2 * \sin\left(\frac{\psi}{2}\right)$$

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Ex:-

نفرض العزم السالب كنسبة من العزم الموجب

$$\sum M\theta_A = \sum\left(M_x * \theta_x + M_y\theta_y\right)_A = (m + i_1 m) * \alpha * L * \frac{1}{\beta L}$$

$$= (1 + i_1) * \frac{m\alpha}{\beta} = (m\alpha + i_1 m\alpha) * \frac{1}{\beta}$$

$$\sum \theta_B = \sum\left(M_x * \theta_x + M_y\theta_y\right) = (m + i_2 m) * \alpha L * \frac{1}{(1 - \beta)L}$$

$$= (1 + i_2) * \frac{m\alpha}{1 - \beta} = (m\alpha L + i_2 m\alpha L) * \frac{1}{(1 - \beta)L}$$

$$\sum M\theta_{A,B} = \alpha m \left(\frac{1 + i_1}{\beta} + \frac{1 + i_2}{1 - \beta}\right) \dots \dots \dots (I)$$

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$$\sum w\delta = \frac{1}{2} * w * \alpha * L^2$$

$$\sum M\theta = \sum w\delta$$

$$\frac{w * \alpha * L^2}{2} = \alpha m \left(\frac{1 + i_1}{\beta} + \frac{1 + i_2}{1 - \beta} \right)$$
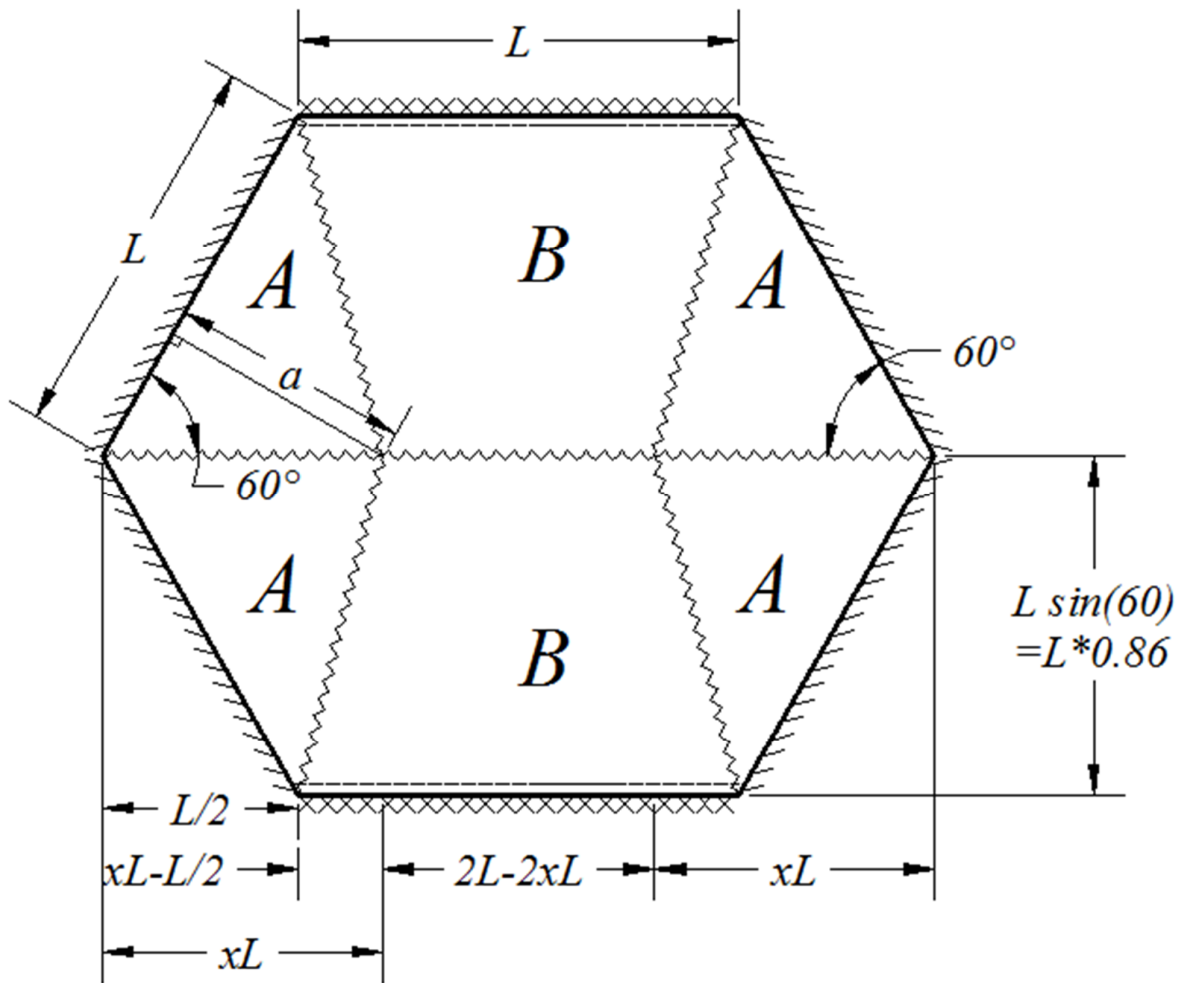
$$IF \ i_1 = i_2$$

$$m = \frac{wL^2}{4} * (\beta - \beta^2)$$

$$\frac{\partial m}{\partial \beta} = 0 \rightarrow \beta = \frac{1}{2}$$

$$\therefore m = \frac{wL^2}{16}$$

Ex9:- isotropic slab $m = -m = +m$

$$a = xL \sin(60) = 0.8 * xL$$

$$\sum M\theta_A = \left(mL * \frac{1}{a}\right) * 4$$

$$\sum M\theta_B = \left[(mL + mL) * \frac{1}{L\sin(60)}\right] * 2$$

$$\sum w\delta = 4\left[w * \frac{La}{2} * \frac{1}{3}\right] + 2\left[(2L - 2xL) * L\sin(60) * w * \frac{1}{2}\right]$$
$$+ 4\left[\left(xL - \frac{L}{2}\right) * \frac{L\sin(60)}{2} * \frac{1}{3}\right]$$

$$\sum w\delta = 4\left[\frac{L * L\sin(60)}{2} * \frac{1}{3} * w\right]$$
$$+ 2[(L^2\sin(60) * xL^2\sin(60))w]$$
$$+ 4\left[\frac{xL^2\sin(60) - \frac{L^2\sin(60)}{2}}{6} * w\right]$$

$$\sum M\theta = \frac{4m}{\sin(60)}\left[\frac{1 + x}{x}\right]$$

$$\sum M\theta = \sum w\delta$$

$$\frac{L^2\sin(60)}{3} * w * (5 - 2x) = \frac{4m}{\sin(60)}\left[\frac{1 + x}{x}\right]$$

$$w = \frac{16m(1 + x)}{L^2(5x - 2x^2)}$$

$$\frac{\partial w}{\partial x} = 0 \rightarrow x = 0.87$$

$$\therefore w = 10.54 \ m/L^2$$

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Ex10:-

A uniformly loaded isotropic reinforced slab is supported as shown in Figure. Using yield line theory to analysis this system determine the collapse load (w_u). The plastic moment of the reinforcement per meter width of slab for top and bottom reinforcement are $(1.5M_n) \& (M_n)$ respectively.
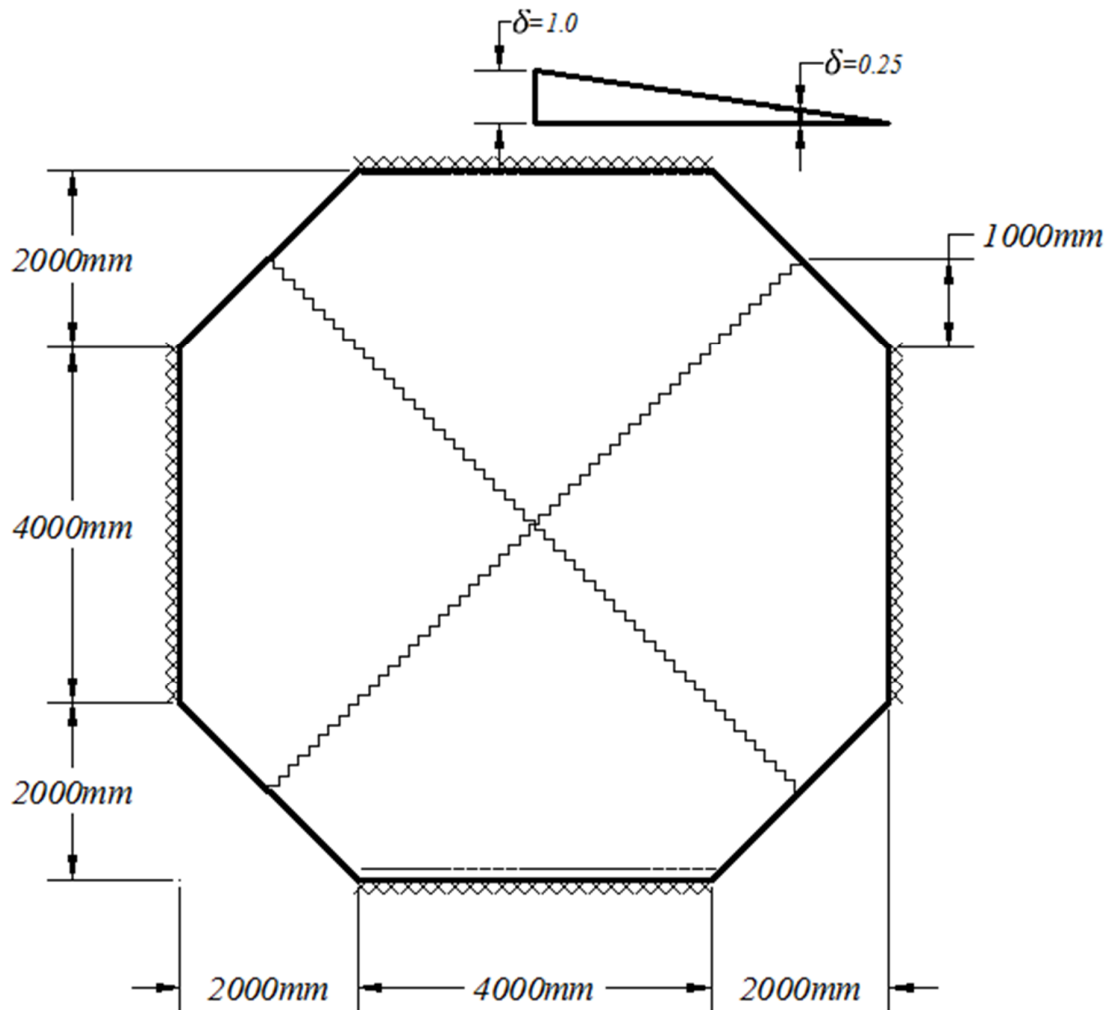
Sol

$$\frac{\delta_1}{\delta_2} = \frac{4}{1}$$

$$\delta_2 = 0.25\delta_1$$

- Exterior work

$$\sum w\delta = volum\ of\ slab$$

$$= w_n \left[\left(8 * 8 * 1 * 1 * \frac{1}{3} \right) - 4 \left(\frac{2*2}{2} * 0.25 * \frac{1}{3} \right) \right]$$

$$= 20.66 w_n$$

- Interior work

$$\sum M\theta = [(M_n * 2 * 3) + (1.5 M_n * 4)] * \frac{1}{4} * 4 = 12 M_n$$

$\sum M\theta = \sum w\delta$

$20.66 w_n = 12 M_n$

$w_n = 0.58 M_n$

Ex11:-

For the R.C isotropic slab shown in Figure find the concentrated collapse load (P) which is applied at the center of slab use $+M = -M$

Sol

Case 1

$$w_e = P(\delta = 1) = P$$

$$w_i = m(l) * \frac{1}{\frac{l}{2}} * 4 = 16\ m$$

$$w_e = w_i \rightarrow P = 16\ m$$
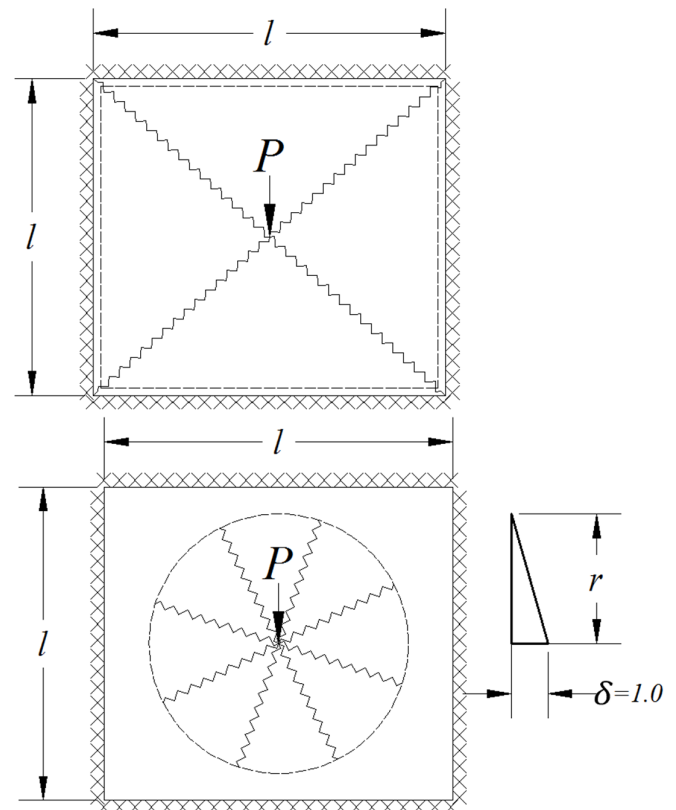
Case 2

$$w_e = P(\delta = 1) = P$$

$$w_i = m(2\pi r) * \frac{1}{r} + m(2\pi r) * \frac{1}{r}$$
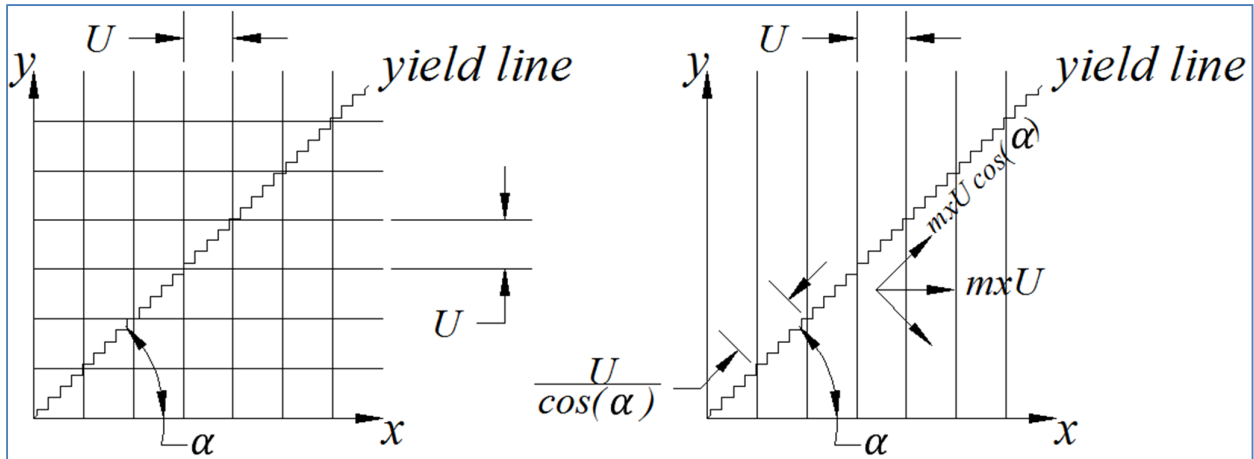
$$= m(4\pi)$$

$$w_e = w_i \rightarrow P = 4m\pi = 12.566\ m$$

$$\therefore P = 12.566\ m$$

Orthotropic reinforcement and skewed yield lines

- Bars in the x-direction are at spacing (U) and have moment resistance m_x per unit length about the x-axis
- The resisting moment per bar about the x-axis is $(m_x U)$ and the component of that resistance about the $\propto$-axis is $(m_x U * \cos(\propto))$
- The resistance moment per unit length along the $\propto$-axis provided by the y-direction bars is therefore

$$m \propto y = \frac{m_x U \cos(\propto)}{4/\cos(\propto)} = m_x \cos^2(\propto)$$

$$m \propto x = m_y \sin^2(\propto)$$

For the combined sets of bars the resisting moment per unit length measured along the $\propto$-axis

$$m_\propto = m_x \cos^2 \propto + m_y \sin^2 \propto$$

For special case where $m_x = m_y = m$

$$m_\propto = m(\cos^2(\propto) + \sin^2(\propto)) = m$$

Plastic Analysis
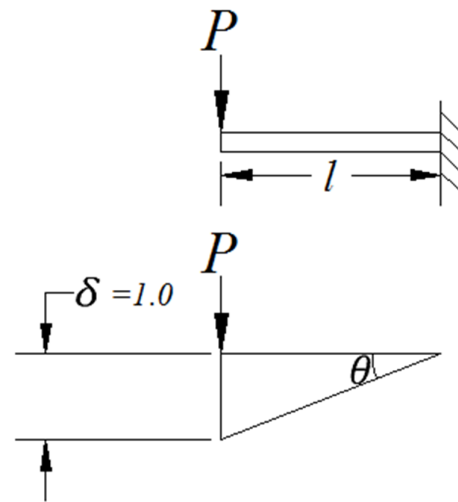
M_P = Plastic moment

Exterior work

$$w_e = P * \delta$$

Internal work

$$w_i = M_P * \theta = M_P * \frac{\delta}{l}$$

$$w_e = w_i$$

$$P * \delta = M_P * \frac{\delta}{l} \rightarrow M_P = P * l$$
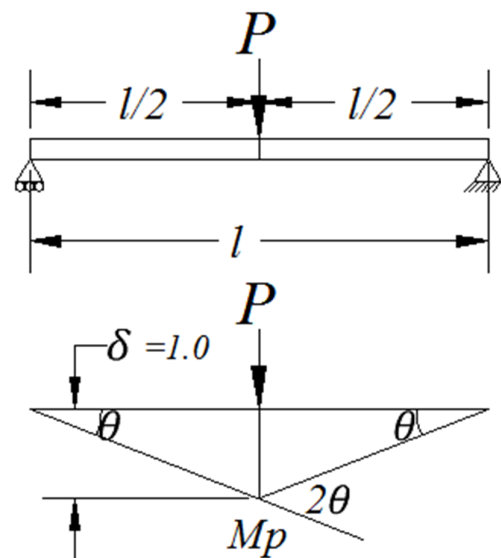
Ex11

$$w_e = P\delta$$

$$w_i = M_P * 2\theta = M_P * \left(\frac{\delta}{l/2}\right) * 2$$

$$= 4M_P * \frac{\delta}{l}$$

$$w_e = w_i$$

$$P\delta = 4M_P * \frac{\delta}{l}$$

$$M_P = \frac{pl}{4}$$
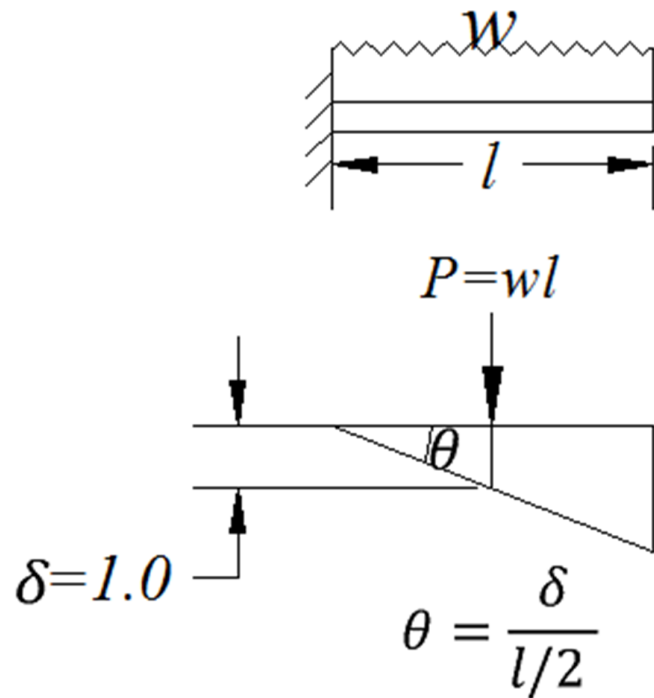
Ex12:-

$$w_e = wl\delta$$

$$w_i = M_P * \theta = M_P \left(\frac{\delta}{l/2}\right)$$

$$= 2M_P * \frac{\delta}{l}$$

$$w_e = w_i$$

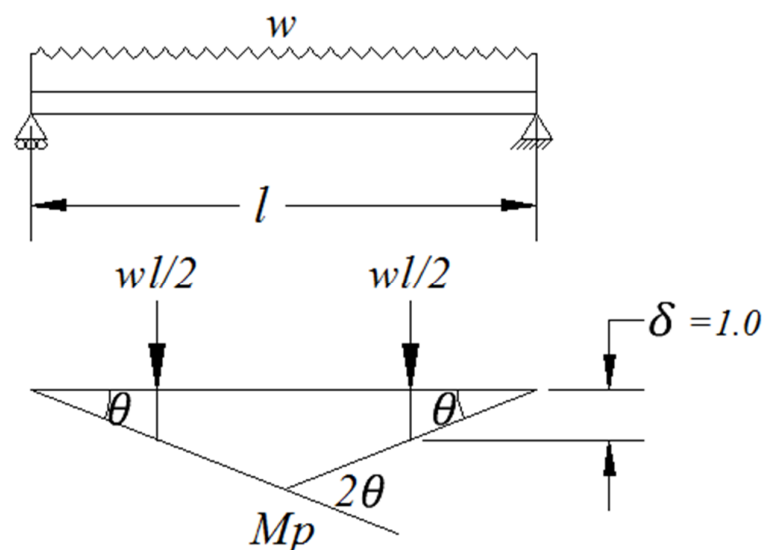$$wl\delta = 2M_P * \frac{\delta}{l}$$

$$M_P = \frac{wl^2}{2}$$

$$P = wl$$

$$\delta = 1.0$$

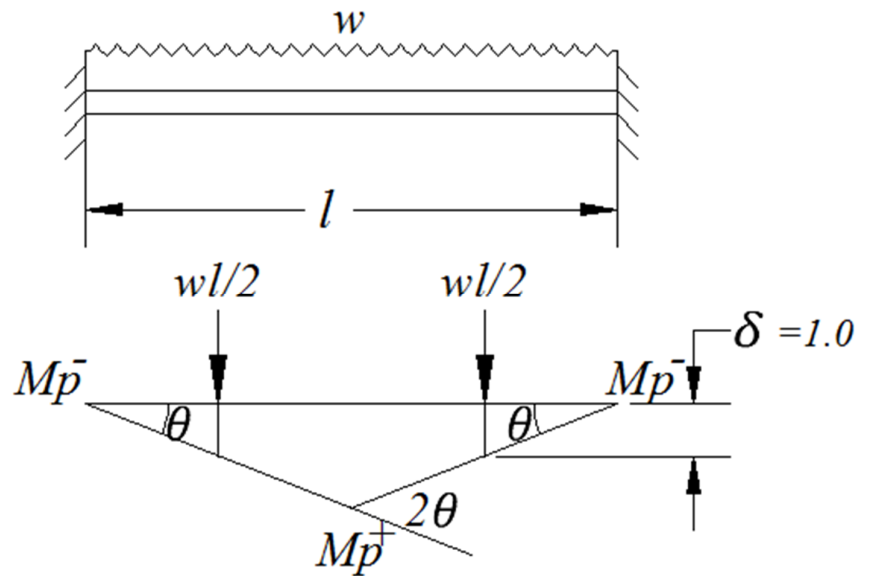$$\theta = \frac{\delta}{l/2}$$

Ex 13

$$w_e = \left(\frac{wl}{2} * \delta\right) * 2$$

$$w_i = M_P * 2\theta = M_P * 2\left(\frac{\delta}{l/4}\right) = 8M_P \frac{\delta}{l}$$

$$M_P = \frac{wl^2}{8}$$

$wl/2$ $wl/2$

$$\delta = 1.0$$

2θ

Mp

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Ex14:

$$w_e = \left(\frac{wl}{2} * \delta\right) * 2$$

w

l

$wl/2$ $wl/2$

$\delta = 1.0$

Mp^- θ θ Mp^-

2θ

Mp^+

$$w_i = M_P^- * \theta + M_P^- * \theta + M_P^+(2\theta) = 4M_P\theta = 4M_P\left(\frac{\delta}{l/4}\right)$$

$$= 16M_P * \frac{\delta}{l}$$

$$w_e = w_i$$

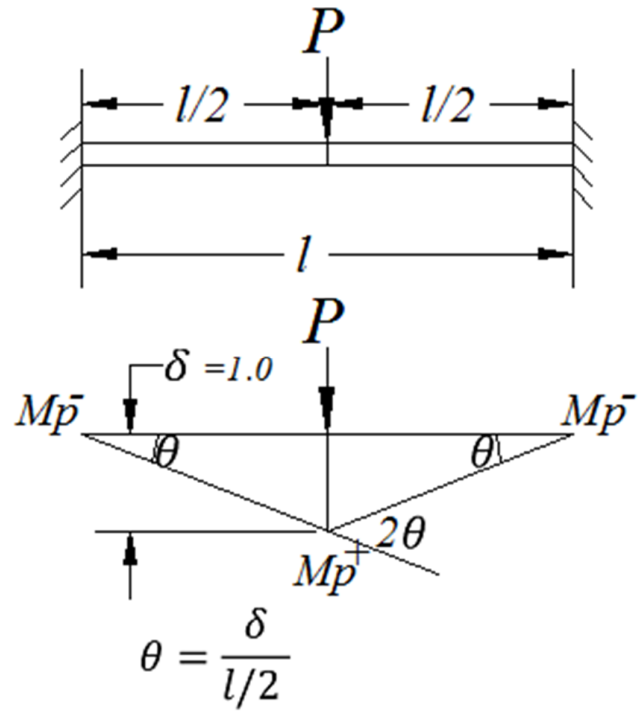$$wl\delta = 16M_P * \frac{\delta}{l}$$

$$M_P = \frac{wl^2}{16}$$

Ex15

$$w_e = P * \delta$$

$$w_i = 4M_P\theta = 4M_P\left(\frac{\delta}{l/2}\right)$$

$$w_e = w_i$$

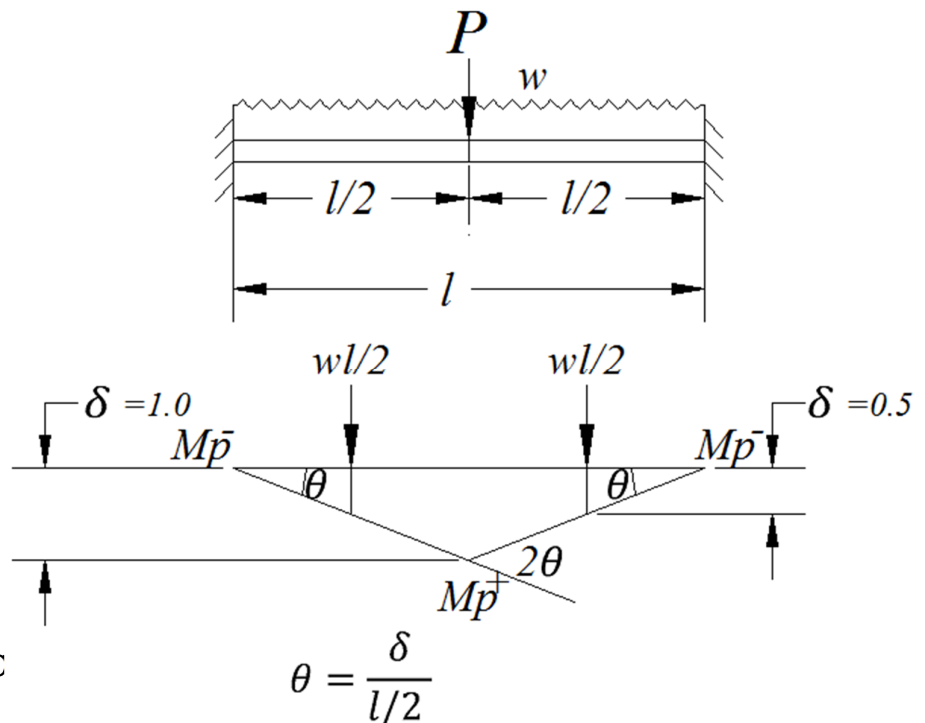$$P * \delta = 8M_P\frac{\delta}{l}$$

$$M_P = \frac{pl}{8}$$

$$\theta = \frac{\delta}{l/2}$$

Ex16

$$w_e = P * \delta + \left(\frac{wl}{2} * \frac{\delta}{2}\right) * 2$$

$$w_i = 4M_P * \frac{\delta}{l/2} = 8M_P\frac{\delta}{l}$$

$$w_e = w_i$$

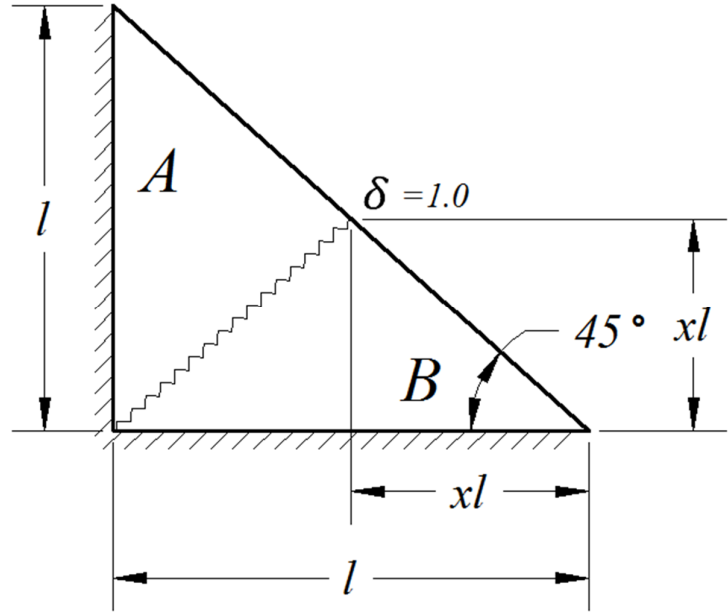$$P * \delta + \left(\frac{wl}{2} * \frac{\delta}{2}\right) * 2 = 8M_P\frac{\delta}{l}$$

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$$\theta = \frac{\delta}{l/2}$$

$$M_P = \frac{wl^2}{16} + \frac{pl}{8}$$

Ex 17

Positive moment = M_n

Negative moment = $2M_n$

Sol

External work

$$\Sigma w\delta = l * \frac{xl}{2} * w * \frac{1}{3} + \frac{l(1-x)l}{2} * \frac{1}{3} * wl = \frac{wl^2}{6} = w * \frac{l*l}{2} * \frac{1}{3}$$

Internal work

$$\Sigma M\theta = [(M_n * xl) + (2M_n * l)] * \frac{1}{l(1-x)} + [M_n * l(1-x)] * \frac{1}{xl}$$

$$= M_n(x+2) * \frac{1}{1-x} + \frac{(1-x)M_n}{x}$$

$$= M_n * \frac{(x^2 + 2x + 1 - 2x + x^2)}{x - x^2}$$

$\Sigma M\theta = \Sigma w\delta$

$$M_n * \frac{(2x^2 + 1)}{x - x^2} = \frac{wl^2}{6}$$

$$w_n = \frac{6M_n}{l^2} * \frac{(2x^2 + 1)}{x - x^2}$$

$$\frac{\partial w}{\partial x} = 0 \rightarrow x = 0.36$$

$$w_n = \frac{32.78}{l^2} M_n$$

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Ex 18

$$\sum w\delta = w\left[L * L * \frac{1}{3}\right] = \frac{wL^2}{3}$$

$$\sum M\theta = 4m^- * L * \frac{1}{L/2} + 4m * L * \frac{1}{L/2} = 8m + 8m = 16m$$

$$\sum w\delta = \sum M\theta$$

$$\frac{wL^2}{3} = 16m \qquad w = \frac{48}{L^2} * m$$

Ex19

$$\sum w\delta = w\left[\left(kL * \frac{L-kL}{2} * 4 * \frac{1}{2}\right) + \left(8 * \frac{\frac{L-kL}{2} * \frac{L-kL}{2}}{2} * \frac{1}{3}\right)\right]$$

$$= w\left[kL^2 - k^2L^2 + \frac{L^2 - 2kL^2 + k^2L^2}{3}\right]$$

$$= w\left[\frac{L^2}{3} - 2 * \frac{kL^2}{3} - \frac{2}{3}k^2L^2 + kL^2\right]$$

$$\sum M\theta = \left[4m^- * L * \frac{1}{\frac{L-kL}{2}}\right] + \left[8m * \frac{L-kL}{2} * \frac{1}{\frac{L-kL}{2}}\right]$$

$$= \frac{8m + 8m - 8mk}{1-k} = m\frac{16 - 8k}{1-k}$$

$$\sum w\delta = \sum M\theta$$

$$w\left[\frac{L^2}{3} - 2 * \frac{kL^2}{3} - \frac{2}{3}k^2L^2 + kL^2\right] = m\frac{16-8k}{1-k}$$

$$w = \frac{3m}{L^2} * \frac{16-8k}{1 - 2k^3 - 3k^2}$$

$$16 + 32k^3 - 48k^2 - 8k = 0$$

$$k(32k^2 - 48k + 8) = 0$$

$$k = 0.191$$

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