Math 1220-1
Lecture Notes

Section 7.8

Certain combinations of e^x and e^{-x} occur frequently in mathematics, and so they are given special names.

Definition 1 ***Hyperbolic Functions***

$$\sinh x = \tfrac{e^x - e^{-x}}{2} \qquad \cosh x = \tfrac{e^x + e^{-x}}{2}$$

$$\tanh x = \tfrac{\sinh x}{\cosh x} \qquad \coth x = \tfrac{\cosh x}{\sinh x}$$

$$\operatorname{sech} x = \tfrac{1}{\cosh x} \qquad \operatorname{csch} x = \tfrac{1}{\sinh x}$$

The hyperbolic functions are closely related to the trigonometric functions. The first example of this follows from the identity

$$\cosh^2 x - \sinh^2 x = 1$$

We can verify this from the definitions,

$$\cosh^2 x - \sinh^2 x = \frac{e^{2x} + e^{-2x} + 2}{4} - \frac{e^{2x} + e^{-2x} - 2}{4} = 1$$

Just like $x = \cos t$ and $y = \sin t$ gives a parametric representation for a circle, the above formula implies that $x = \cosh t$ and $y = \sinh t$ gives a parametric representation for the right branch of the hyperbola $x^2 - y^2 = 1$. Just like their trigonometric counterparts, $\sinh x$, $\tanh x$, $\operatorname{csch} x$, and $\coth x$ are odd functions, $\cosh x$ and $\operatorname{sech} x$ are even functions. This is also easily verified from the definitions.

Definition 2 ***Derivatives of Hyperbolic Functions***

$$\begin{array}{ll} D_x \sinh x = \cosh x & D_x \cosh x = \sinh x \\ D_x \tanh x = \operatorname{sech}^2 x & D_x \coth x = -\operatorname{csch}^2 x \\ D_x \operatorname{sech} x = -\operatorname{sech} x \tanh x & D_x \operatorname{csch} x = -\operatorname{csch} x \coth x \end{array}$$

Another way the trigonometric and hyperbolic functions are connected concerns differential equations. The functions $\sin x$ and $\cos x$ are solutions to the differential equation $y'' + y = 0$, while $\sinh x$ and $\cosh x$ are solutions to the differential equation $y'' - y = 0$.

Example 1 *Find $D_x \cosh(\tan x)$.*

Solution

$$\begin{aligned} D_x \cosh(\tan x) &= \sinh(\tan x) D_x(\tan x) \\ &= \sec^2 x \cdot \sinh(\tan x) \end{aligned}$$

Example 2 *Find* $\int \coth x \, dx$.

Solution Let $u = \sinh x$, so that $du = \cosh x \, dx$.

$$\begin{aligned} \int \coth x \, dx &= \int \tfrac{\cosh x}{\sinh x} \, dx &= \int \frac{1}{u} \, du \\ &= \ln |u| + C &= \ln |\sinh x| + C \end{aligned}$$

Since hyperbolic sine and hyperbolic tangent have positive derivatives, they are increasing functions and automatically have inverses. To obtain inverses for hyperbolic cosine and hyperbolic secant, we restrict their domains to $x \geq 0$. Thus,

$$\begin{aligned} x = \sinh^{-1} y &\Leftrightarrow y = \sinh x \\ x = \cosh^{-1} y &\Leftrightarrow y = \cosh x \text{ and } x \geq 0 \\ x = \tanh^{-1} y &\Leftrightarrow y = \tanh x \\ x = \operatorname{sech}^{-1} y &\Leftrightarrow y = \operatorname{sech} x \text{ and } x \geq 0 \end{aligned}$$

Since the hyperbolic functions are defined in terms of exponentials, it seems likely that the inverse hyperbolic functions can be defined in terms of the natural log, and indeed this is so as long as we make some necessary domain restrictions.

$$\begin{aligned} \sinh^{-1} x &= \ln \left(x + \sqrt{x^2 + 1} \right) \\ \cosh^{-1} x &= \ln \left(x + \sqrt{x^2 - 1} \right), \quad x \geq 1 \\ \tanh^{-1} x &= \tfrac{1}{2} \ln \tfrac{1+x}{1-x}, \quad -1 < x < 1 \\ \operatorname{sech}^{-1} x &= \ln \left(\tfrac{1 + \sqrt{1 - x^2}}{x} \right) \end{aligned}$$

Each of the above functions is differentiable, so we get

$$\begin{aligned} D_x \sinh^{-1} x &= \tfrac{1}{\sqrt{x^2+1}} \\ D_x \cosh^{-1} x &= \tfrac{1}{\sqrt{x^2-1}}, \quad x > 1 \\ D_x \tanh^{-1} x &= \tfrac{1}{1-x^2}, \quad -1 < x < 1 \\ D_x \operatorname{sech}^{-1} x &= \tfrac{-1}{x\sqrt{1-x^2}}, \quad 0 < x < 1 \end{aligned}$$

Example 3 *Find* $D_x \cosh^{-1} \left(x^3 \right)$.

Solution

$$\begin{aligned} D_x \cosh^{-1} \left(x^3 \right) &= \frac{1}{\sqrt{\left(x^3 \right)^2 - 1}} D_x x^3 \\ &= \frac{3x^2}{\sqrt{x^6 - 1}} \end{aligned}$$